Hydrostatic compression of a elastic sphere

This text is integrating part of the homonymous link in PEEI: a computer program for the numerical solution of systems of partial differential equations.

System of measurement: International System of Units, with the exception of the force that is expressed in $N\times10^{-12}$.

Coordinate system: Cartesian

Coordinates: $\underline{x}$ of which: $\underline{x}\equiv\{x_i;i=1,3\}$ $[x_i]\equiv$[length] $\underline{\Re}\langle x_i\rangle\equiv(-\infty,\infty)$, $\underline{x}$ a point of the deformed medium.

Coordinate versors: $\{\mathbf{\upsilon}_i;i=1,3\}$

Unknown functions: $\{s_1,s_2,s_3,\tau_{11},\tau_{12},\tau_{13},\tau_{22},\tau_{23},\tau_{33}\}$ of which: $s_i=x_i-X_i$, $[s_i]\equiv$[length], $\underline{X}\equiv\{X_i;i=1,3\}$, $\underline{X}$ the position of the point $\underline{x}$ in the undeformed medium, $\mathbf{s}\equiv\Sigma_{i=1,3}(s_i\cdot\mathbf{\upsilon}_i)$, $\mathbf{s}$ the displacement of the point $\underline{X}$, $\{\tau_{11},\tau_{12},\tau_{13},\tau_{22},\tau_{23},\tau_{33}\}$ the six independent components of the stress tensor, $[\tau_{ij}]\equiv$[stress], $\tau_{ij}=\tau_{ji}$.

Differential analytical model:

$\partial\tau_{11}(\underline{x})/\partial x_1+\partial\tau_{12}(\underline{x})/\partial x_2+\partial\tau_{13}(\underline{x})/\partial x_3+F_1(\underline{x})=0$

$\partial\tau_{12}(\underline{x})/\partial x_1+\partial\tau_{22}(\underline{x})/\partial x_2+\partial\tau_{23}(\underline{x})/\partial x_3+F_2(\underline{x})=0$

$\partial\tau_{13}(\underline{x})/\partial x_1+\partial\tau_{23}(\underline{x})/\partial x_2+\partial\tau_{33}(\underline{x})/\partial x_3+F_3(\underline{x})=0$

$\{(1+\nu)\cdot\tau_{ij}(\underline{x})-\delta_{ij}\cdot\nu\cdot(\tau_{11}(\underline{x})+\tau_{22}(\underline{x})+\tau_{33}(\underline{x}))-E\cdot(\partial s_i(\underline{x})/\partial x_j+\partial s_j(\underline{x})/\partial x_i)/2=0;j=i,3;i=1,3\}$

of which: $\mathbf{F}\equiv\Sigma_{i=1,3}(F_i\cdot\mathbf{\upsilon}_i)$, $\mathbf{F}$ the body force per unit volume, $\{\delta_{ij}=0;\forall i\neq j\}$ $\{\delta_{ij}=1;\forall i=j\}$, E Young's modulus, ν Poisson's ratio, $E=0.21$ $\nu=0.3$.

Related relations:

$$\varepsilon_{ij}(\underline{x})=(\partial s_i(\underline{x})/\partial x_j+\partial s_j(\underline{x})/\partial x_i)/2=(1+\nu)\cdot\tau_{ij}(\underline{x})/E-\delta_{ij}\cdot\nu\cdot(\tau_{11}(\underline{x})+\tau_{22}(\underline{x})+\tau_{33}(\underline{x}))/E \quad (1)$$

$$\omega_{ij}(\underline{x})=(\partial s_i(\underline{x})/\partial x_j-\partial s_j(\underline{x})/\partial x_i)/2$$

$$s_i(\underline{x}_B)=s_i(\underline{x}_A)+\Sigma_{j=1,3}(\omega_{ij}(\underline{x}_A)\cdot(x_{Bj}-x_{Aj}))+\int_{A,B}(\Theta_i(c)\cdot dc) \quad (2)$$

$$\Theta_i(c)\equiv\Sigma_{j=1,3}(\varepsilon_{ij}(\underline{x}(c))\cdot x_j'(c)+(x_{Bj}-x_j(c))\cdot\Sigma_{k=1,3}((\partial\varepsilon_{ik}(\underline{x}(c))/\partial x_j-\partial\varepsilon_{jk}(\underline{x}(c))/\partial x_i)\cdot x_k'(c))) \quad (3)$$

of which here, $\varepsilon_{ij}=\varepsilon_{ji}$, $\underline{x}(c)\equiv\{x_i(c);i=1,3\}$ $\underline{x}_A\equiv\{x_{Ai};i=1,3\}=\underline{x}(A)$ $\underline{x}_B\equiv\{x_{Bi};i=1,3\}=\underline{x}(B)$.

Definition set: $\{\underline{x}\,/\,x_1^2+x_2^2+x_3^2\leq R^2\}$ $R=1/2$.

Conditions:

$F_1(\underline{x})=F_2(\underline{x})=F_3(\underline{x})=0$ $\{s_i(\underline{x}_A)=0;i=1,3\}$ $\partial s_1(\underline{x}_A)/\partial x_2=\partial s_1(\underline{x}_A)/\partial x_3=\partial s_2(\underline{x}_A)/\partial x_3=0$ $\underline{x}_A\equiv\{0,0,0\}$ (4)

$\{\tau_{11}=\tau_{22}=\tau_{33}=-\mathbb{P},\tau_{12}=\tau_{13}=\tau_{23}=0;\forall x_1^2+x_2^2+x_3^2=R^2\}$ $\mathbb{P}=0.1$ (5)

In PEEI executions, the (5) is applied on the surface of the body that approximates the sphere. This is coherent with the validity of this script for a body of every shape.

Related files: mad.txt

Exact solution:

The (5) and (1) imply

$\varepsilon_{ij}(\underline{x})=\delta_{ij}\cdot(2\cdot\nu-1)\cdot\mathbb{P}/E$ (6)

From these, (1), $\partial s_i(\underline{x})/\partial x_j=\varepsilon_{ij}(\underline{x})+\omega_{ij}(\underline{x})$ and (4) follows $\omega_{ij}(\underline{x}_A)=0$. This, $\{s_i(\underline{x}_A)=0;i=1,3\}$ and (2) imply

$s_i(\underline{x}_B)=\int_{A,B}(\Theta_i(c)\cdot dc)$ (7)

Are placed

$\int_{A,B}(\Theta_i(c)\cdot dc)=\int_{A,P}(\Theta_i(c)\cdot dc)+\int_{P,Q}(\Theta_i(c)\cdot dc)+\int_{Q,B}(\Theta_i(c)\cdot dc)$ $\underline{x}(P)\equiv\{x_{B1},0,0\}$ $\underline{x}(Q)\equiv\{x_{B1},x_{B2},0\}$ $\{x_2'(c)=x_3'(c)=0,x_1'(c)=1;\forall c\in[A,P]\}$ $\{x_1'(c)=x_3'(c)=0,x_2'(c)=1;\forall c\in[P,Q]\}$ $\{x_1'(c)=x_2'(c)=0,x_3'(c)=1;\forall c\in[Q,B]\}$ (8)

These, (3) and (6) imply

$\{\Theta_1(c)=(2\cdot\nu-1)\cdot\mathbb{P}/E,\Theta_2(c)=\Theta_3(c)=0;\forall c\in[A,P]\}$ $\{\Theta_2(c)=(2\cdot\nu-1)\cdot\mathbb{P}/E,\Theta_1(c)=\Theta_3(c)=0;\forall c\in[P,Q]\}$ $\{\Theta_3(c)=(2\cdot\nu-1)\cdot\mathbb{P}/E,\Theta_1(c)=\Theta_2(c)=0;\forall c\in[Q,B]\}$

From these, (7) and (8) follows

$s_i(\underline{x})=(2\cdot\nu-1)\cdot\mathbb{P}\cdot x_i/E$ (9)

and hence $\mathbf{s}=\mathbf{s}(\underline{x})\equiv\Sigma_{i=1,3}(s_i(\underline{x})\cdot\boldsymbol{\upsilon}_i)=(2\cdot\nu-1)\cdot\mathbb{P}\cdot\Sigma_{i=1,3}(x_i\cdot\boldsymbol{\upsilon}_i)/E$. This imply $\mathbf{s}(\underline{x})=(2\cdot\nu-1)\cdot\mathbb{P}\cdot\mathbf{r}(\underline{x})/E$ where $\mathbf{r}(\underline{x})\equiv\Sigma_{i=1,3}(x_i\cdot\boldsymbol{\upsilon}_i)$; i.e. $\mathbf{s}$ is radial, is directed towards $\underline{x}_A$, and have magnitude $r(\underline{x})\cdot(1-2\cdot\nu)\cdot\mathbb{P}/E$ of which $r(\underline{x})=(\Sigma_{i=1,3}(x_i^2))^{0.5}$. The (9) is valid for the hydrostatic compression of an elastic body of every shape.

Note: In the following diagrams, the symbols + (plus), □ (empty square) and ■ (full square) are respectively inherent to $\underline{x}$, $\underline{X}$ determined by means of $X_i=x_i-s_i$ and (9), and $\underline{X}$ determined by means of $X_i=x_i-s_i$ where s_i is calculated by PEEI.

Case 5: points-5.txt, mem-5.bin, cond-5.txt, sol-5.txt, plot-5-1.jpg, plot-5-2.jpg, plot-5-3.jpg, plot-5-4.jpg

Case 7: points-7.txt, mem-7.bin, cond-7.txt, sol-7.txt, plot-7-1.jpg, plot-7-2.jpg, plot-7-3.jpg, plot-7-4.jpg

Case 9: points-9.txt, mem-9.bin, cond-9.txt, sol-9.txt, plot-9-1.jpg, plot-9-2.jpg, plot-9-3.jpg, plot-9-4.jpg

Case 11: points-11.txt, mem-11.bin, cond-11.txt, sol-11.txt, plot-11-1.jpg, plot-11-2.jpg, plot-11-3.jpg, plot-11-4.jpg

Bibliography:

[1] YU. A. AMENZADE, *Theory of Elasticity*, Mir Publishers, 1979, Moscow