Hagen-Poiseuille Flow

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http://www.giacomo.lorenzoni.name/PEEI_4.0.0.1/Hagen-Poiseuille_flow/

http://www.giacomo.lorenzoni.name/PEEI_4.0.0.1/PEEIapplDown.aspx?var=13

Hagen-Poiseuille flow

This text is integrating part of the homonymous link in PEEI: a computer program for the numerical solution of systems of partial differential equations.

System of measurement: International System of Units

Coordinate system: Cylindrical

Coordinates: $\underline{c}$ of which $\underline{c}\equiv\{c_i;i=1,3\}$ $\mathfrak{R}\langle c_1\rangle\equiv[0,\infty)$ $\mathfrak{R}\langle c_2\rangle\equiv[0,2\cdot\pi)$ $\mathfrak{R}\langle c_3\rangle\equiv(-\infty,\infty)$

Coordinate versors: $\{\boldsymbol{\kappa}_i;i=1,3\}$

Unknown functions: $\{w_1,w_2,w_3,P\}$ of which $\mathbf{w}\equiv\Sigma_{i=1,3}(w_i\cdot\boldsymbol{\kappa}_i)$, $[w_i]\equiv$[speed], $\mathbf{w}$ the velocity vector, $[P]\equiv$[pressure].

Deduction of the differential analytical model:

The $c_1=c_1(\underline{x})\equiv(x_1^2+x_2^2)^{0.5}$ $c_2=c_2(\underline{x})\equiv\arcsin(x_2/(x_1^2+x_2^2)^{0.5})=\arctan\langle x_2/x_1\rangle+K$ $c_3=c_3(\underline{x})\equiv x_3$ of which $\underline{x}\equiv\{x_i;i=1,3\}$ $\mathfrak{R}\langle x_i\rangle\equiv(-\infty,\infty)$, imply the $x_1=x_1(\underline{c})\equiv c_1\cdot\cos\langle c_2\rangle$ $x_2=x_2(\underline{c})\equiv c_1\cdot\sin\langle c_2\rangle$ $x_3=x_3(\underline{c})\equiv c_3$, where $\underline{x}$ are the coordinates of a Cartesian coordinate system that have the versors $\{\boldsymbol{\upsilon}_i;i=1,3\}$, and K is a constant determined by the trigonometric quadrant of $\{x_1,x_2\}$. Hence

$$\partial c_1/\partial x_1=\cos\langle c_2\rangle=x_1/c_1\ \partial c_1/\partial x_2=\sin\langle c_2\rangle=x_2/c_1\ \partial c_2/\partial x_1=-\sin\langle c_2\rangle/c_1=-x_2/c_1^2$$
$$\partial c_2/\partial x_2=\cos\langle c_2\rangle/c_1=x_1/c_1^2\ \partial c_1/\partial x_3=\partial c_2/\partial x_3=\partial c_3/\partial x_1=\partial c_3/\partial x_2=0\ \partial c_3/\partial x_3=1$$

$$\partial_{hi}\equiv\partial c_h/\partial x_i=\hat{o}_{h1i3}\cdot\hat{o}_{h2i3}\cdot\hat{o}_{h3i1}\cdot\hat{o}_{h3i2}\cdot((\breve{o}_{h1i1}\cdot x_1+\breve{o}_{h1i2}\cdot x_2)/c_1+(\breve{o}_{h2i2}\cdot x_1-\breve{o}_{h2i1}\cdot x_2)/c_1^2+\breve{o}_{h3i3})$$

$$\partial^2{}_{hij}\equiv\partial^2 c_h/\partial x_i\partial x_j=\hat{o}_{h1i3}\cdot\hat{o}_{h1j3}\cdot\hat{o}_{h2i3}\cdot\hat{o}_{h2j3}\cdot\hat{o}_{h3i1}\cdot\hat{o}_{h3j1}\cdot\hat{o}_{h3i2}\cdot\hat{o}_{h3j2}\cdot$$
$$(\delta_{j1}\cdot(\breve{o}_{h1i1}/c_1-(\breve{o}_{h1i1}\cdot x_1+\breve{o}_{h1i2}\cdot x_2)\cdot x_1/c_1^3+\breve{o}_{h2i2}/c_1^2-2\cdot(\breve{o}_{h2i2}\cdot x_1-\breve{o}_{h2i1}\cdot x_2)\cdot x_1/c_1^4)+$$
$$\delta_{j2}\cdot(\breve{o}_{h1i2}/c_1-(\breve{o}_{h1i1}\cdot x_1+\breve{o}_{h1i2}\cdot x_2)\cdot x_2/c_1^3-\breve{o}_{h2i1}/c_1^2-2\cdot(\breve{o}_{h2i2}\cdot x_1-\breve{o}_{h2i1}\cdot x_2)\cdot x_2/c_1^4))$$

where $\breve{o}_{ijhk}=1-\hat{o}_{ijhk}$, $\hat{o}_{ijhk}=0$ if $i=j$ $h=k$ and otherwise $\hat{o}_{ijhk}=1$.

The $\boldsymbol{\kappa}_i\equiv\Sigma_{i=1,3}(\kappa_{ii}\cdot\boldsymbol{\upsilon}_i)$ implies $\boldsymbol{\kappa}_i\cdot\boldsymbol{\upsilon}_j=\Sigma_{i=1,3}(\kappa_{ii}\cdot\boldsymbol{\upsilon}_i\cdot\boldsymbol{\upsilon}_j)$. This and $\boldsymbol{\upsilon}_i\cdot\boldsymbol{\upsilon}_j=\delta_{ij}$ (where $\{\delta_{ij}=0;\forall i\neq j\}$ $\{\delta_{ij}=1;\forall i=j\}$), imply $\boldsymbol{\kappa}_i\cdot\boldsymbol{\upsilon}_j=\kappa_{ij}$ and then $\kappa_{ij}=\cos\langle\alpha_{ij}\rangle$ where α_{ij} is the angle between $\boldsymbol{\kappa}_i$ and $\boldsymbol{\upsilon}_j$. Hence

$$\kappa_{11}=\kappa_{22}=x_1/c_1\ \kappa_{33}=1\ \kappa_{12}=-\kappa_{21}=x_2/c_1\ \kappa_{13}=\kappa_{23}=\kappa_{31}=\kappa_{32}=0$$

$$\kappa_{ij}=\hat{o}_{i1j3}\cdot\hat{o}_{i2j3}\cdot\hat{o}_{i3j1}\cdot\hat{o}_{i3j2}\cdot((\breve{o}_{i1j1}+\breve{o}_{i2j2})\cdot x_1/c_1+(\breve{o}_{i1j2}-\breve{o}_{i2j1})\cdot x_2/c_1+\breve{o}_{i3j3})$$

$$\partial_{ijh}\equiv\partial\kappa_{ij}/\partial x_h=\hat{o}_{i1j3}\cdot\hat{o}_{i2j3}\cdot\hat{o}_{i3j1}\cdot\hat{o}_{i3j2}\cdot(\delta_{h1}\cdot((\breve{o}_{i1j1}+\breve{o}_{i2j2})/c_1-((\breve{o}_{i1j1}+\breve{o}_{i2j2})\cdot x_1+(\breve{o}_{i1j2}-\breve{o}_{i2j1})\cdot x_2)\cdot x_1/c_1^3)+$$
$$\delta_{h2}\cdot((\breve{o}_{i1j2}-\breve{o}_{i2j1})/c_1-((\breve{o}_{i1j1}+\breve{o}_{i2j2})\cdot x_1+(\breve{o}_{i1j2}-\breve{o}_{i2j1})\cdot x_2)\cdot x_2/c_1^3))$$

$\partial^2_{ijhk} \equiv \partial^2 \kappa_{ij}/\partial x_h \partial x_k = -\hat{o}_{i1j3} \cdot \hat{o}_{i2j3} \cdot \hat{o}_{i3j1} \cdot \hat{o}_{i3j2} \cdot (\delta_{k1} \cdot (\delta_{h1} \cdot (2 \cdot (\breve{o}_{i1j1} + \breve{o}_{i2j2}) \cdot x_1/c_1^3 +$
$((\breve{o}_{i1j1} + \breve{o}_{i2j2}) \cdot x_1 + (\breve{o}_{i1j2} - \breve{o}_{i2j1}) \cdot x_2) \cdot (c_1^{-3} - 3 \cdot x_1^2/c_1^5)) +$
$\delta_{h2} \cdot ((\breve{o}_{i1j2} - \breve{o}_{i2j1}) \cdot x_1/c_1^3 + (\breve{o}_{i1j1} + \breve{o}_{i2j2}) \cdot x_2/c_1^3 - 3 \cdot ((\breve{o}_{i1j1} + \breve{o}_{i2j2}) \cdot x_1 + (\breve{o}_{i1j2} - \breve{o}_{i2j1}) \cdot x_2) \cdot x_1 \cdot x_2/c_1^5)) +$
$\delta_{k2} \cdot (\delta_{h1} \cdot ((\breve{o}_{i1j1} + \breve{o}_{i2j2}) \cdot x_2/c_1^3 + (\breve{o}_{i1j2} - \breve{o}_{i2j1}) \cdot x_1/c_1^3 - 3 \cdot ((\breve{o}_{i1j1} + \breve{o}_{i2j2}) \cdot x_1 + (\breve{o}_{i1j2} - \breve{o}_{i2j1}) \cdot x_2) \cdot x_1 \cdot x_2/c_1^5) +$
$\delta_{h2} \cdot (2 \cdot (\breve{o}_{i1j2} - \breve{o}_{i2j1}) \cdot x_2/c_1^3 + ((\breve{o}_{i1j1} + \breve{o}_{i2j2}) \cdot x_1 + (\breve{o}_{i1j2} - \breve{o}_{i2j1}) \cdot x_2) \cdot (c_1^{-3} - 3 \cdot x_2^2/c_1^5))))$

The $\boldsymbol{\upsilon}_i \cdot \boldsymbol{\upsilon}_j = \delta_{ij}$ $\boldsymbol{\kappa}_i \cdot \boldsymbol{\upsilon}_j = \kappa_{ij}$ $\Sigma_{j=1,3}(v_j \cdot \boldsymbol{\kappa}_j) = \Sigma_{j=1,3}(v_j \cdot \boldsymbol{\upsilon}_j)$, imply

$v_i = \Sigma_{j=1,3}(v_j \cdot \kappa_{ji})$ (1)

From $F(\underline{x}) \equiv F(\underline{c}(\underline{x}))$ where $\underline{c}(\underline{x}) \equiv \{c_i(\underline{x}); i=1,3\}$, follows

$\partial F(\underline{x})/\partial x_i = \partial F(\underline{c}(\underline{x}))/\partial x_i = \Sigma_{i=1,3}((\partial F(\underline{c}(\underline{x}))/\partial c_i) \cdot (\partial c_i(\underline{x})/\partial x_i)) \equiv \Sigma_{i=1,3}((\partial F/\partial c_i) \cdot \partial_{ii})$ (2)

$\partial^2 F(\underline{x})/\partial x_i \partial x_j = \partial^2 F(\underline{c}(\underline{x}))/\partial x_i \partial x_j = \partial(\partial F(\underline{c}(\underline{x}))/\partial x_i)/\partial x_j = \Sigma_{i=1,3}(\partial((\partial F(\underline{c}(\underline{x}))/\partial c_i) \cdot (\partial c_i(\underline{x})/\partial x_i))/\partial x_j) =$
$\Sigma_{i=1,3}((\partial F(\underline{c}(\underline{x}))/\partial c_i) \cdot (\partial^2 c_i(\underline{x})/\partial x_i \partial x_j) + (\partial c_i(\underline{x})/\partial x_i) \cdot \Sigma_{j=1,3}((\partial^2 F(\underline{c}(\underline{x}))/\partial c_i \partial c_j) \cdot (\partial c_j(\underline{x})/\partial x_j))) \equiv$
$\Sigma_{i=1,3}((\partial F/\partial c_i) \cdot \partial^2_{iij} + \Sigma_{j=1,3}(\partial_{ii} \cdot \partial_{jj} \cdot (\partial^2 F/\partial c_i \partial c_j)))$ (3)

The continuity equation for incompressible fluids and the stationary incompressible Navier-Stokes equations for constant viscosity, in Cartesian coordinates are respectively

$\Sigma_{i=1,3}(\partial w_i/\partial x_i) = 0$ (4)

$\{\rho \cdot (\Sigma_{j=1,3}(w_j \cdot (\partial w_i/\partial x_j)) - F_i) + \partial P/\partial x_i - \mu \cdot \Sigma_{j=1,3}(\Sigma_{h=1,3}(\Sigma_{k=1,3}(\delta_{jikh} \cdot (\partial^2 w_k/\partial x_h \partial x_j)))) = 0; i=1,3\}$ (5)

of which: $\mathbf{w} = \Sigma_{i=1,3}(w_i \cdot \boldsymbol{\upsilon}_i)$, $[w_i] \equiv$[speed], $[\rho] \equiv$[density], $\mathbf{F} \equiv \Sigma_{i=1,3}(F_i \cdot \boldsymbol{\upsilon}_i)$, $[F_i] \equiv$[force/mass], $\mathbf{F}$ the body force vector per unit mass, $[\mu] \equiv$[dynamic viscosity], $\delta_{ijkh} \equiv \delta_{ik} \cdot \delta_{jh} + \delta_{jk} \cdot \delta_{ih} - (2/3) \cdot \delta_{hk} \cdot \delta_{ij}$.

From $\Sigma_{i=1,3}(w_i \cdot \boldsymbol{\upsilon}_i) = \Sigma_{i=1,3}(w_i \cdot \boldsymbol{\kappa}_i)$ and (1) follows $w_i = \Sigma_{j=1,3}(w_j \cdot \kappa_{ji})$. This, (4) (5) (2) and (3), imply (6) and (7).

Differential analytical model:

$\Sigma_{i=1,3}(\Sigma_{j=1,3}(\partial_{jii} \cdot w_j + \Sigma_{h=1,3}(\kappa_{ji} \cdot \partial_{hi} \cdot (\partial w_j/\partial c_h)))) = 0$ (6)

$\{\Sigma_{j=1,3}(\partial_{ji} \cdot (\partial P/\partial c_j) +$
$\Sigma_{h=1,3}(\Sigma_{k=1,3}(\rho \cdot \kappa_{hj} \cdot \partial_{kij} \cdot w_h \cdot w_k +$
$\Sigma_{m=1,3}(\rho \cdot \kappa_{hj} \cdot \kappa_{ki} \cdot \partial_{mj} \cdot w_h \cdot (\partial w_k/\partial c_m) - \mu \cdot \delta_{jikh} \cdot \partial^2_{mkhj} \cdot w_m -$
$\mu \cdot \delta_{jikh} \cdot \Sigma_{n=1,3}((\partial_{nj} \cdot \partial_{mkh} + \partial_{nh} \cdot \partial_{mkj} + \kappa_{mk} \cdot \partial^2_{nhj}) \cdot (\partial w_m/\partial c_n) +$
$\Sigma_{p=1,3}(\kappa_{mk} \cdot \partial_{nh} \cdot \partial_{pj} \cdot (\partial^2 w_m/\partial c_n \partial c_p)))))))) - \rho \cdot F_i = 0; i=1,3\}$ (7)

of which $\rho = 998.2071$ $\mu = 0.001003$. The (6) and (7) are respectively, in cylindrical coordinates, the continuity equation for incompressible fluids and the stationary incompressible Navier-Stokes equations for constant viscosity.

Known functions: $\{F_i; i=1,156\}$ of which $\{F_i \equiv F_i; i=1,3\}$ $F_1 = F_3 = 0$ $F_2 = -9.80665$ $F_{A\langle h,i\rangle} = \partial_{hi}$ $F_{B\langle h,i,j\rangle} = \partial^2_{hij}$ $F_{C\langle i,j\rangle} = \kappa_{ij}$ $F_{D\langle i,j,h\rangle} = \partial_{ijh}$ $F_{E\langle i,j,h,k\rangle} = \partial^2_{ijhk}$ $A_{hi} = 3 + h + 3 \cdot (i-1)$ $B_{hij} = A_{33} + h + 3 \cdot (i-1) + 9 \cdot (j-1)$ $C_{ij} = B_{333} + i + 3 \cdot (j-1)$ $D_{ijh} = C_{33} + i + 3 \cdot (j-1) + 9 \cdot (h-1)$ $E_{ijhk} = D_{333} + i + 3 \cdot (j-1) + 9 \cdot (h-1) + 27 \cdot (k-1)$.

Definition set: $\{\underline{c} / R_1 \le c_1 \le R_2; 0 \le c_2 < 2 \cdot \pi; 0 \le c_3 \le L_3\}$ $R_1 = 0.001$ $R_2 = 10$ $L_3 = 1000$.

Conditions: $w_1(\underline{c})=w_2(\underline{c})=\partial w_3/\partial c_2=\partial w_3/\partial c_3=0$ $w_3(R_2,c_2,c_3)=0$ $\partial P(\underline{c})/\partial c_3=K=-1$ $P(R_2,0,0)=100000$ (8)

Related files: mad.txt

Exact solution: From here follows

$w_3=w_3(c_1)\equiv K\cdot(c_1^2-R_2^2)/(4\cdot\mu)$ (9)

Note: In the following diagrams, the continuous line and the symbol ● (full circle) are respectively inherent to (9) and solution calculated by PEEI.

Case 1: points-1.txt, points-1.bin, cond-1.txt, sol-1.txt, plot-1.jpg

Case 2: points-2.txt, points-2.bin, cond-2.txt, sol-2.txt, plot-2.jpg

Case 3: points-3.txt, points-3.bin, cond-3.txt, sol-3.txt, plot-3.jpg

Case 4: points-4.txt, points-4.bin, cond-4.txt, sol-4.txt, plot-4.jpg

Case 5: points-5.txt, points-5.bin, cond-5.txt, sol-5.txt, plot-5.jpg

Case 6: points-6.txt, points-6.bin, cond-6.txt, sol-6.txt, plot-6.jpg

Case 7: points-7.txt, points-7.bin, cond-7.txt, sol-7.txt, plot-7.jpg

Case 8: points-8.txt, points-8.bin, cond-8.txt, sol-8.txt, plot-8.jpg

Case 9: points-9.txt, points-9.bin, cond-9.txt, sol-9.txt, plot-9.jpg

Case 10: points-10.txt, points-10.bin, cond-10.txt, sol-10.txt, plot-10.jpg

Case 11: points-11.txt, points-11.bin, cond-11.txt, sol-11.txt, plot-11.jpg

Case 12: points-12.txt, points-12.bin, cond-12.txt, sol-12.txt, plot-12.jpg

Case 13: points-13.txt, points-13.bin, cond-13.txt, sol-13.txt, plot-13.jpg

Case 14: points-14.txt, points-14.bin, cond-14.txt, sol-14.txt, plot-14.jpg

Case 15: points-15.txt, points-15.bin, cond-15.txt, sol-15.txt, plot-15.jpg

Case 16: points-16.txt, points-16.bin, cond-16.txt, sol-16.txt, plot-16.jpg

Case 17: points-17.txt, points-17.bin, cond-17.txt, sol-17.txt, plot-17.jpg

Case 18: points-18.txt, points-18.bin, cond-18.txt, sol-18.txt, plot-18.jpg

Case 19: points-19.txt, points-19.bin, cond-19.txt, sol-19.txt, plot-19.jpg

Case 20: points-20.txt, points-20.bin, cond-20.txt, sol-20.txt, plot-20.jpg

Case 21: points-25.txt, points-25.bin, cond-25.txt, sol-25.txt, plot-25.jpg

Case 22: points-30.txt, points-30.bin, cond-30.txt, sol-30.txt, plot-30.jpg

Case 23: points-35.txt, points-35.bin, cond-35.txt, sol-35.txt, plot-35.jpg

Case 24: points-40.txt, points-40.bin, cond-40.txt, sol-40.txt, plot-40.jpg

Case 25: points-45.txt, points-45.bin, cond-45.txt, sol-45.txt, plot-45.jpg

Case 26: points-50.txt, points-50.bin, cond-50.txt, sol-50.txt, plot-50.jpg

Case 27: points-60.txt, points-60.bin, cond-60.txt, sol-60.txt, plot-60.jpg