Plane Poiseuille Flow

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Giacomo Lorenzoni

http://www.giacomo.lorenzoni.name/PEEI_4.0.0.1/Plane_Poiseuille_flow/

http://www.giacomo.lorenzoni.name/PEEI_4.0.0.1/PEEIapplDown.aspx?var=10

Plane Poiseuille flow

This text is integrating part of the homonymous link in PEEI: a computer program for the numerical solution of systems of partial differential equations.

System of measurement: International System of Units.

Coordinate system: Cartesian

Coordinates: $\underline{x}$ of which $\underline{x}\equiv\{x_i;i=1,3\}$ $[x_i]\equiv$[length] $\underline{\Re}\langle x_i\rangle\equiv(-\infty,\infty)$

Coordinate versors: $\{\boldsymbol{\upsilon}_i;i=1,3\}$

Unknown functions: $\{W_1,W_2,W_3,P\}$ of which $\mathbf{W}\equiv\Sigma_{i=1,3}(W_i\cdot\boldsymbol{\upsilon}_i)$, $[W_i]\equiv$[speed], $\mathbf{W}$ the velocity vector, $[P]\equiv$[pressure].

Differential analytical model:

$$\Sigma_{i=1,3}(\partial W_i/\partial x_i)=0 \quad (1)$$

$$\{\rho\cdot(\Sigma_{j=1,3}(W_j\cdot(\partial W_i/\partial x_j))-F_i)+\partial P/\partial x_i-\mu\cdot\Sigma_{j=1,3}(\Sigma_{h=1,3}(\Sigma_{k=1,3}(\delta_{jikh}\cdot(\partial^2 W_k/\partial x_h\partial x_j))))=0;i=1,3\} \quad (2)$$

of which: $\rho=998.2071$ $[\rho]\equiv$[density], $\mathbf{F}\equiv\Sigma_{i=1,3}(F_i\cdot\boldsymbol{\upsilon}_i)$, $[F_i]\equiv$[force/mass], $\mathbf{F}$ the body force vector per unit mass, $\mu=0.001003$ $[\mu]\equiv$[dynamic viscosity], $\delta_{ijkh}\equiv\delta_{ik}\cdot\delta_{jh}+\delta_{jk}\cdot\delta_{ih}-(2/3)\cdot\delta_{hk}\cdot\delta_{ij}$, $\{\delta_{ij}=0;\forall i\neq j\}$ $\{\delta_{ij}=1;\forall i=j\}$. The (1) is the continuity equation for incompressible fluids, the (2) are the stationary incompressible Navier-Stokes equations for constant viscosity.

Definition set: $\{\underline{x}\ /\ 0\leq x_1\leq L_1;0\leq x_2\leq L_2;0\leq x_3\leq L_3\}$ $L_1=1$ $L_2=1000$ $L_3=0.1$.

Conditions:

$$F_1=F_2=0 \quad F_3=-9.80665 \quad W_1(\underline{x})=W_3(\underline{x})=W_2(x_1,x_2,0)=W_2(x_1,x_2,L_3)=\partial W_2(\underline{x})/\partial x_1=0 \quad \partial P(\underline{x})/\partial x_2=K=-1 \quad P(0,0,L_3)=1000000 \quad (3)$$

Related files: mad.txt

Exact solution:

From (1) (2) and (3) follows $\partial W_2(\underline{x})/\partial x_2=\partial P(\underline{x})/\partial x_1=0$, $\mu\cdot(\partial^2 W_2(\underline{x})/\partial x_3^2)=K$, $\partial P(\underline{x})/\partial x_3=\rho\cdot F_3$, and then

$$dP(\underline{x})\equiv\Sigma_{i=1,3}((\partial P(\underline{x})/\partial x_i)\cdot dx_i)=K\cdot dx_2+\rho\cdot F_3\cdot dx_3 \quad (4)$$

Are placed

$A\leq c\leq B$ $\underline{x}(c)\equiv\{x_i(c);i=1,3\}$ $\underline{x}_A\equiv\{0,0,L_3\}\equiv\{x_{Ai};i=1,3\}=\underline{x}(A)$ $\underline{x}_B\equiv\{x_{Bi};i=1,3\}=\underline{x}(B)$

These and (4) imply $dP(\underline{x}(c))=K\cdot dx_2(c)+\rho\cdot F_3\cdot dx_3(c)=(K\cdot x_2'(c)+\rho\cdot F_3\cdot x_3'(c))\cdot dc$ and then

$P(\underline{x}_B)=P(\underline{x}_A)+\int_{A,B}((K\cdot x_2'(c)+\rho\cdot F_3\cdot x_3'(c))\cdot dc)$ (5)

Are placed

$\int_{A,B}(f(c)\cdot dc)=\int_{A,N}(f(c)\cdot dc)+\int_{N,Q}(f(c)\cdot dc)+\int_{Q,B}(f(c)\cdot dc)$ $\underline{x}(N)\equiv\{0,0,x_{B3}\}$ $\underline{x}(Q)\equiv\{x_{B1},0,x_{B3}\}$
$\{x_1'(c)=x_2'(c)=0,x_3'(c)=-1;\forall c\in[A,N]\}$ $\{x_2'(c)=x_3'(c)=0,x_1'(c)=1;\forall c\in[N,Q]\}$
$\{x_1'(c)=x_3'(c)=0,x_2'(c)=1;\forall c\in[Q,B]\}$ (6)

that imply $\int_{A,B}((K\cdot x_2'(c)+\rho\cdot F_3\cdot x_3'(c))\cdot dc)=K\cdot x_{B2}+\rho\cdot F_3\cdot(x_{B3}-L_3)$. From this and (5) follows

$P(\underline{x})=P(\underline{x}_A)+K\cdot x_2+\rho\cdot F_3\cdot(x_3-L_3)$ (7)

The $\mu\cdot(\partial^2 w_2(\underline{x})/\partial x_3^2)-K=0$ implies $\mu\cdot(\partial w_2(\underline{x})/\partial x_3)-K\cdot x_3+R(x_1,x_2)=0$, and then $\mu\cdot w_2(\underline{x})-K\cdot x_3^2/2+R(x_1,x_2)\cdot x_3=0$. This and $w_2(x_1,x_2,L_3)=0$ imply $R(x_1,x_2)=K\cdot L_3/2$. Hence

$w_2(\underline{x})=K\cdot x_3\cdot(x_3-L_3)/(2\cdot\mu)$ (8)

Note: In the following diagrams, the symbols □ (empty square) and ● (full circle) are respectively inherent to the solution expressed by (7) and (8), and the solution calculated by PEEI.

Case 1: points-1.txt, mem-1.bin, cond-1.txt, sol-1.txt, plot-1-1.jpg, plot-1-2.jpg

Case 2: points-2.txt, mem-2.bin, cond-2.txt, sol-2.txt, plot-2-1.jpg, plot-2-2.jpg

Case 3: points-3.txt, mem-3.bin, cond-3.txt, sol-3.txt, plot-3-1.jpg, plot-3-2.jpg

Case 4: points-4.txt, mem-4.bin, cond-4.txt, sol-4.txt, plot-4-1.jpg, plot-4-2.jpg

Case 5: points-5.txt, mem-5.bin, cond-5.txt, sol-5.txt, plot-5-1.jpg, plot-5-2.jpg

Case 6: points-6.txt, mem-6.bin, cond-6.txt, sol-6.txt, plot-6-1.jpg, plot-6-2.jpg

Case 7: points-7.txt, mem-7.bin, cond-7.txt, sol-7.txt, plot-7-1.jpg, plot-7-2.jpg

Case 8: points-8.txt, mem-8.bin, cond-8.txt, sol-8.txt, plot-8-1.jpg, plot-8-2.jpg

Case 9: points-9.txt, mem-9.bin, cond-9.txt, sol-9.txt, plot-9-1.jpg, plot-9-2.jpg

Case 10: points-10.txt, mem-10.bin, cond-10.txt, sol-10.txt, plot-10-1.jpg, plot-10-2.jpg

Case 11: points-11.txt, mem-11.bin, cond-11.txt, sol-11.txt, plot-11-1.jpg, plot-11-2.jpg

Case 12: points-12.txt, mem-12.bin, cond-12.txt, sol-12.txt, plot-12-1.jpg, plot-12-2.jpg

Case 13: points-13.txt, mem-13.bin, cond-13.txt, sol-13.txt, plot-13-1.jpg, plot-13-2.jpg

Case 14: points-14.txt, mem-14.bin, cond-14.txt, sol-14.txt, plot-14-1.jpg, plot-14-2.jpg

Case 15: points-15.txt, mem-15.bin, cond-15.txt, sol-15.txt, plot-15-1.jpg, plot-15-2.jpg

Case 16: points-16.txt, mem-16.bin, cond-16.txt, sol-16.txt, plot-16-1.jpg, plot-16-2.jpg

Case 17: points-17.txt, mem-17.bin, cond-17.txt, sol-17.txt, plot-17-1.jpg, plot-17-2.jpg