Pure Elastic Bending of a Prismatic Bar

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http://www.giacomo.lorenzoni.name/PEEI_4.0.0.1/Pure_elastic_bending_of_a_prismatic_bar/

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Pure elastic bending of a prismatic bar

This text is integrating part of the homonymous link in PEEI: a computer program for the numerical solution of systems of partial differential equations.

System of measurement: International System of Units, with the exception of the force that is expressed in $N \times 10^{-12}$.

Coordinate system: Cartesian

Coordinates: $\underline{x}$ of which: $\underline{x} \equiv \{x_i; i=1,3\}$ $[x_i] \equiv$ [length] $\Re\langle x_i \rangle \equiv (-\infty, \infty)$, $\underline{x}$ a point of the deformed medium.

Coordinate versors: $\{\boldsymbol{\upsilon}_i; i=1,3\}$

Unknown functions: $\{s_1, s_2, s_3, \tau_{11}, \tau_{12}, \tau_{13}, \tau_{22}, \tau_{23}, \tau_{33}\}$ of which: $s_i = x_i - X_i$, $[s_i] \equiv$ [length], $\underline{X} \equiv \{X_i; i=1,3\}$, $\underline{X}$ the position of the point $\underline{x}$ in the undeformed medium, $\mathbf{s} \equiv \Sigma_{i=1,3}(s_i \cdot \boldsymbol{\upsilon}_i)$, $\mathbf{s}$ the displacement of the point $\underline{X}$, $\{\tau_{11}, \tau_{12}, \tau_{13}, \tau_{22}, \tau_{23}, \tau_{33}\}$ the six independent components of the stress tensor, $[\tau_{ij}] \equiv$ [stress], $\tau_{ij} = \tau_{ji}$.

Differential analytical model:

$\partial\tau_{11}(\underline{x})/\partial x_1 + \partial\tau_{12}(\underline{x})/\partial x_2 + \partial\tau_{13}(\underline{x})/\partial x_3 + F_1(\underline{x}) = 0$

$\partial\tau_{12}(\underline{x})/\partial x_1 + \partial\tau_{22}(\underline{x})/\partial x_2 + \partial\tau_{23}(\underline{x})/\partial x_3 + F_2(\underline{x}) = 0$

$\partial\tau_{13}(\underline{x})/\partial x_1 + \partial\tau_{23}(\underline{x})/\partial x_2 + \partial\tau_{33}(\underline{x})/\partial x_3 + F_3(\underline{x}) = 0$

$\{(1+\nu) \cdot \tau_{ij}(\underline{x}) - \delta_{ij} \cdot \nu \cdot (\tau_{11}(\underline{x}) + \tau_{22}(\underline{x}) + \tau_{33}(\underline{x})) - E \cdot (\partial s_i(\underline{x})/\partial x_j + \partial s_j(\underline{x})/\partial x_i)/2 = 0; j=i,3; i=1,3\}$

of which: $\mathbf{F} \equiv \Sigma_{i=1,3}(F_i \cdot \boldsymbol{\upsilon}_i)$, $\mathbf{F}$ the body force per unit volume, $\{\delta_{ij}=0; \forall i \neq j\}$ $\{\delta_{ij}=1; \forall i=j\}$, E Young's modulus, ν Poisson's ratio, $E=0.21$ $\nu=0.3$.

Related relations:

$\varepsilon_{ij}(\underline{x}) = (\partial s_i(\underline{x})/\partial x_j + \partial s_j(\underline{x})/\partial x_i)/2 = (1+\nu) \cdot \tau_{ij}(\underline{x})/E - \delta_{ij} \cdot \nu \cdot (\tau_{11}(\underline{x}) + \tau_{22}(\underline{x}) + \tau_{33}(\underline{x}))/E$ (1)

$\omega_{ij}(\underline{x}) = (\partial s_i(\underline{x})/\partial x_j - \partial s_j(\underline{x})/\partial x_i)/2$ (2)

$T_i(\underline{x}) = \Sigma_{j=1,3}(\tau_{ji}(\underline{x}) \cdot \check{n}_j(\underline{x}))$ (3)

$s_i(\underline{x}_B) = s_i(\underline{x}_A) + \Sigma_{j=1,3}(\omega_{ij}(\underline{x}_A) \cdot (x_{Bj} - x_{Aj})) + \int_{A,B}(\Theta_i(c) \cdot dc)$ (4)

$\Theta_i(c) \equiv \Sigma_{j=1,3}(\varepsilon_{ij}(\underline{x}(c)) \cdot x_j'(c) + (x_{Bj} - x_j(c)) \cdot \Sigma_{k=1,3}((\partial\varepsilon_{ik}(\underline{x}(c))/\partial x_j - \partial\varepsilon_{jk}(\underline{x}(c))/\partial x_i) \cdot x_k'(c)))$ (5)

of which here, $\varepsilon_{ij} = \varepsilon_{ji}$, $\mathbf{T}(\underline{x}) \equiv \Sigma_{i=1,3}(T_i(\underline{x}) \cdot \boldsymbol{\upsilon}_i)$ $\check{\mathbf{n}}(\underline{x}) \equiv \Sigma_{i=1,3}(\check{n}_i(\underline{x}) \cdot \boldsymbol{\upsilon}_i)$, $\mathbf{T}(\underline{x})$ the stress vector in a point of a plane with normal outward versor $\check{\mathbf{n}}(\underline{x})$, $\underline{x}(c) \equiv \{x_i(c); i=1,3\}$ $\underline{x}_A \equiv \{x_{Ai}; i=1,3\} = \underline{x}(A)$ $\underline{x}_B \equiv \{x_{Bi}; i=1,3\} = \underline{x}(B)$

Definition set: $\{\underline{x}\ /\ 0\le x_1\le L_1;0\le x_2\le L_2;0\le x_3\le L_3\}$ $L_1=1$ $L_2=10$ $L_3=10$.

Conditions:

$$F_1(\underline{x})=F_2(\underline{x})=F_3(\underline{x})=0\ \{s_i(\underline{x}_A)=0;i=1,3\}\ \partial s_1(\underline{x}_A)/\partial x_2=\partial s_1(\underline{x}_A)/\partial x_3=\partial s_2(\underline{x}_A)/\partial x_3=0 \qquad (6)$$

where $\underline{x}_A\equiv\{L_1/2,0,L_3/2\}$.

$T_1(0,x_2,x_3)=T_2(0,x_2,x_3)=T_3(0,x_2,x_3)=T_1(L_1,x_2,x_3)=T_2(L_1,x_2,x_3)=T_3(L_1,x_2,x_3)=T_1(x_1,x_2,0)=T_2(x_1,x_2,0)=T_3(x_1,x_2,0)=T_1(x_1,x_2,L_3)=T_2(x_1,x_2,L_3)=T_3(x_1,x_2,L_3)=T_1(x_1,0,x_3)=T_3(x_1,0,x_3)=T_1(x_1,L_2,x_3)=T_3(x_1,L_2,x_3)=0$ $T_2(x_1,0,x_3)=-\mathbb{P}\cdot((2/L_3)\cdot x_3-1)$ $T_2(x_1,L_2,x_3)=\mathbb{P}\cdot((2/L_3)\cdot x_3-1)$ $\mathbb{P}=0.1$

From these and (3) follows

$\tau_{11}(0,x_2,x_3)=\tau_{12}(0,x_2,x_3)=\tau_{13}(0,x_2,x_3)=\tau_{11}(L_1,x_2,x_3)=\tau_{12}(L_1,x_2,x_3)=\tau_{13}(L_1,x_2,x_3)=\tau_{13}(x_1,x_2,0)=\tau_{23}(x_1,x_2,0)=\tau_{33}(x_1,x_2,0)=\tau_{13}(x_1,x_2,L_3)=\tau_{23}(x_1,x_2,L_3)=\tau_{33}(x_1,x_2,L_3)=\tau_{12}(x_1,0,x_3)=\tau_{23}(x_1,0,x_3)=\tau_{12}(x_1,L_2,x_3)=\tau_{23}(x_1,L_2,x_3)=0$ $\tau_{22}(x_1,0,x_3)=\tau_{22}(x_1,L_2,x_3)=\mathbb{P}\cdot((2/L_3)\cdot x_3-1)$

Related files: mad.txt

Exact solution:

From previous conditions follows $\tau_{11}(\underline{x})=\tau_{12}(\underline{x})=\tau_{13}(\underline{x})=\tau_{33}(\underline{x})=\tau_{23}(\underline{x})=0$ $\tau_{22}(\underline{x})=\mathbb{P}\cdot((2/L_3)\cdot x_3-1)$. These and (1) imply

$$\varepsilon_{12}(\underline{x})=\varepsilon_{13}(\underline{x})=\varepsilon_{23}(\underline{x})=0\ \varepsilon_{11}(\underline{x})=\varepsilon_{33}(\underline{x})=-\nu\cdot\mathbb{P}\cdot((2/L_3)\cdot x_3-1)/E\ \varepsilon_{22}(\underline{x})=\mathbb{P}\cdot((2/L_3)\cdot x_3-1)/E \qquad (7)$$

From these, (1), (2), $\partial s_i(\underline{x})/\partial x_j=\varepsilon_{ij}(\underline{x})+\omega_{ij}(\underline{x})$ and (6) follows $\omega_{ij}(\underline{x}_A)=0$. This, $\{s_i(\underline{x}_A)=0;i=1,3\}$ and (4) imply

$$s_i(\underline{x}_B)=\int_{A,B}(\Theta_i(c)\cdot dc) \qquad (8)$$

of which $\underline{x}(A)\equiv\underline{x}_A$.

Are placed

$$\int_{A,B}(\Theta_i(c)\cdot dc)=\int_{A,P}(\Theta_i(c)\cdot dc)+\int_{P,Q}(\Theta_i(c)\cdot dc)+\int_{Q,B}(\Theta_i(c)\cdot dc)\quad \underline{x}(P)\equiv\{L_1/2,x_2,L_3/2\}\quad \underline{x}(Q)\equiv\{x_1,x_2,L_3/2\}\quad \underline{x}(B)\equiv\{x_1,x_2,x_3\}\quad \{x_1'(c)=x_3'(c)=0,x_2'(c)=1;\forall c\in[A,P]\}\quad \{x_2'(c)=x_3'(c)=0,x_1'(c)=1;\forall c\in[P,Q]\}\quad \{x_1'(c)=x_2'(c)=0,x_3'(c)=1;\forall c\in[Q,B]\} \qquad (9)$$

These, (5) and (7) imply

$$\{\Theta_1(c)=0,\Theta_2(c)=\mathbb{P}\cdot((2/L_3)\cdot x_{B3}-1)/E,\Theta_3(c)=-\mathbb{P}\cdot(2/L_3)\cdot(x_{B2}-x_2(c))/E;\forall c\in[A,P]\}$$
$$\{\Theta_1(c)=-\nu\cdot\mathbb{P}\cdot(x_{B3}\cdot(2/L_3)-1)/E,\Theta_2(c)=0,\Theta_3(c)=\nu\cdot\mathbb{P}\cdot(2/L_3)\cdot(x_{B1}-x_1(c))/E;\forall c\in[P,Q]\}$$
$$\{\Theta_1(c)=0,\Theta_2(c)=0,\Theta_3(c)=-\nu\cdot\mathbb{P}\cdot((2/L_3)\cdot x_3(c)-1)/E;\forall c\in[Q,B]\}$$

From these, (8) and (9) follows

$$s_1(\underline{x})=-\nu\cdot\mathbb{P}\cdot(x_3\cdot(2/L_3)-1)\cdot(x_1-L_1/2)/E\ \ s_2(\underline{x})=\mathbb{P}\cdot((2/L_3)\cdot x_3-1)\cdot x_2/E$$
$$s_3(\underline{x})=\mathbb{P}\cdot(\nu\cdot(x_1-L_1/2)^2-x_2^2-\nu\cdot(L_3^2/4+x_3^2-L_3\cdot x_3))/(L_3\cdot E) \qquad (10)$$

Note: In the following diagrams, the symbols + (plus), □ (empty square) and ■ (full square) are respectively inherent to $\underline{x}$, $\underline{X}$ determined by means of $X_i = x_i - s_i$ and (10), and $\underline{X}$ determined by means of $X_i = x_i - s_i$ where s_i is calculated by PEEI.

Case 3-3-3: points-3-3-3.txt, mem-3-3-3.bin, cond-3-3-3.txt, sol-3-3-3.txt, plot-3-3-3-1.jpg, plot-3-3-3-2.jpg, plot-3-3-3-3.jpg

Case 3-5-3: points-3-5-3.txt, mem-3-5-3.bin, cond-3-5-3.txt, sol-3-5-3.txt, plot-3-5-3-1.jpg, plot-3-5-3-2.jpg, plot-3-5-3-3.jpg

Case 3-7-3: points-3-7-3.txt, mem-3-7-3.bin, cond-3-7-3.txt, sol-3-7-3.txt, plot-3-7-3-1.jpg, plot-3-7-3-2.jpg, plot-3-7-3-3.jpg

Case 3-9-3: points-3-9-3.txt, mem-3-9-3.bin, cond-3-9-3.txt, sol-3-9-3.txt, plot-3-9-3-1.jpg, plot-3-9-3-2.jpg, plot-3-9-3-3.jpg

Case 5-3-5: points-5-3-5.txt, mem-5-3-5.bin, cond-5-3-5.txt, sol-5-3-5.txt, plot-5-3-5-1.jpg, plot-5-3-5-2.jpg, plot-5-3-5-3.jpg

Case 5-5-5: points-5-5-5.txt, mem-5-5-5.bin, cond-5-5-5.txt, sol-5-5-5.txt, plot-5-5-5-1.jpg, plot-5-5-5-2.jpg, plot-5-5-5-3.jpg

Case 5-7-5: points-5-7-5.txt, mem-5-7-5.bin, cond-5-7-5.txt, sol-5-7-5.txt, plot-5-7-5-1.jpg, plot-5-7-5-2.jpg, plot-5-7-5-3.jpg

Case 5-9-5: points-5-9-5.txt, mem-5-9-5.bin, cond-5-9-5.txt, sol-5-9-5.txt, plot-5-9-5-1.jpg, plot-5-9-5-2.jpg, plot-5-9-5-3.jpg

Case 7-3-7: points-7-3-7.txt, mem-7-3-7.bin, cond-7-3-7.txt, sol-7-3-7.txt, plot-7-3-7-1.jpg, plot-7-3-7-2.jpg, plot-7-3-7-3.jpg

Case 7-5-7: points-7-5-7.txt, mem-7-5-7.bin, cond-7-5-7.txt, sol-7-5-7.txt, plot-7-5-7-1.jpg, plot-7-5-7-2.jpg, plot-7-5-7-3.jpg

Case 7-7-7: points-7-7-7.txt, mem-7-7-7.bin, cond-7-7-7.txt, sol-7-7-7.txt, plot-7-7-7-1.jpg, plot-7-7-7-2.jpg, plot-7-7-7-3.jpg

Case 7-9-7: points-7-9-7.txt, mem-7-9-7.bin, cond-7-9-7.txt, sol-7-9-7.txt, plot-7-9-7-1.jpg, plot-7-9-7-2.jpg, plot-7-9-7-3.jpg

Bibliography:

[1] YU. A. AMENZADE, *Theory of Elasticity*, Mir Publishers, 1979, Moscow