A method to numerically solve every differential analytical model

Giacomo Lorenzoni

Abstract

A differential analytical model is a system of so many PDEs in the same (or lesser) number of unknown functions where each PDE is of any order and can be nonlinear. Usually the operative interest of such a model is to determine an its solution when is subject to additional conditions like those boundary or initial.

This work exposes the mathematical basis of a program (freeware in http://www.giacomo.lorenzoni.name/peei/) to numerically solve every differential analytical model with every set of additional conditions. In particular is exposed what follows.

Are described the analytical properties of two well known models to approximate a function: the interpolating polynomial and the cubic spline. The values of a natural cubic spline and of its derivatives, in the interpolation nodes, are expressed as linear combinations of the known values of the function to be interpolated and whose coefficients depend only on the nodes. Are obtained new bounds for the errors of a cubic spline. Are presented essential aspects of a curve in a multidimensional Euclidean space, in order to obtain an upper bound for the absolute maximum value of a derivative defined on a curve. Is shown the expression of a partial derivative as a linear combination of directional derivatives and is deduced its optimal approximation. Is formulated the expression of the generic differential analytical model, is identified the main impediment to knowledge of an its exact solution in not knowing its partial derivatives, is circumstantiated the context of information contingently available and is showed how, solving an inherent system of nonlinear equations, can be calculated an its numerical solution. Is exposed an original algorithm that, in this system of nonlinear equations, expresses a derivative as a linear combination of unknowns.

Key words: numerical solution of PDEs systems, differential analytical models, numerical differentiation, splines, graph algorithms, interpolation
MSC: 35G61, 65M22, 65D25, 65D07, 05C85, 65D05
1 Introduction

This paper concerns the generic differential analytical model $M$, intended as a
system of so many equations in the same or lesser number of unknowns where each
equation is a PDE (partial differential equation) of any order and generally nonlinear,
and where the unknowns are functions of same independent variables.

The unknown functions of $M$ are its exact solution. A discrete solution of $M$ is
constituted by a limited number of values that its unknown functions assume in their
domain of definition. A numerical solution is an approximation of a discrete solution.

In relation to $M$, usually is had the operating purpose of determine an its numerical
solution when it is specifically subjected to additional conditions such as those initial
or boundary. So in particular the objective of what follows is to determine a numerical
solution of $M$ when is subject to any set of additional conditions.

The symbology of a technical writing aims to make the comprehension univocal
and shorten the exposure. Therefore, with reference to section 2 of [1] (this work is
also available at http://www.giacomo.lorenzoni.name/arganprobstat/), are premised the
following definitions.
A same object has some names each of which gives to it some properties. An \( A \equiv B \) states that \( A \) and \( B \) are two names of a same object. In identifying the members of an expression, each “\( \equiv \)" is considered at last coherently with the parentheses (and analogously “\( \neq \)”, “\( = \)”, “\( \not\)”). Is intended \( A(B) \equiv A_{\bar{B}}, \Lambda \equiv \text{AND \: \equiv \: “conjunction”}, \lor \equiv \text{OR \: \equiv \: “inclusive disjunction”}, \vee \equiv \text{XOR \: \equiv \: “exclusive disjunction”}. \) Being \( \mathcal{P}, \mathcal{P}_A \) and \( \mathcal{P}_B \) three propositions, \( \neg \mathcal{P} \) is the proposition true if \( \mathcal{P} \) is false and false if \( \mathcal{P} \) is true,

\[
\mathcal{P}_A \Rightarrow \mathcal{P}_B \equiv \mathcal{P}_B \Leftarrow \mathcal{P}_A \equiv \text{“from } \mathcal{P}_A \text{ follows } \mathcal{P}_B” \equiv \text{“} \mathcal{P}_A \text{ entails } \mathcal{P}_B” \equiv \text{“} \mathcal{P}_A \text{ show } \mathcal{P}_B” \equiv \\
\text{“} \mathcal{P}_A \text{ gives rise to } \mathcal{P}_B” \equiv \text{“} \mathcal{P}_A \text{ implies } \mathcal{P}_B” \equiv \text{“} \mathcal{P}_B \text{ is due to } \mathcal{P}_A” \equiv \text{“} \mathcal{P}_B \text{ is obtainable from } \mathcal{P}_A” \\
\{ \mathcal{P}_A \Rightarrow \mathcal{P}_B \} \land \{ \mathcal{P}_B \Leftarrow \mathcal{P}_A \} \equiv \{ \mathcal{P}_A \equiv \mathcal{P}_B \}
\]

\[\{ \mathcal{P}_A \parallel \mathcal{P}_B \} \equiv \text{“} \mathcal{P}_A \text{ subject to condition } \mathcal{P}_B” \equiv \text{“} \mathcal{P}_A \text{ of which } \mathcal{P}_B” \equiv \text{“} \mathcal{P}_A \text{ where } \mathcal{P}_B” \]

\[\mathcal{E}(A \parallel B \parallel C) \equiv \text{“the being } A \text{ a specification of } B \text{ of which } C”\]

where “\( \parallel C” \) may be absent causing so the absence of “of which \( C” \).

Also it is understood \( \text{IPM} \equiv \text{“the first member of”} \) and

\[
\{ \text{from: } A_1; A_2; \ldots; A_i; \text{ follows } B_0 \circ_1 B_1 \circ_2 B_2 \cdots \circ_i B_i \circ_{i+1} B_{i+1} \cdots \circ_{i+j} B_{i+j} \}\equiv \\
\{ A_1 \Rightarrow \{ B_0 \circ_1 B_1 \}; A_2 \Rightarrow \{ B_1 \circ_2 B_2 \}; \ldots; A_i \Rightarrow \{ B_{i-1} \circ_i B_i \}\}
\]

where: each of \( \{ \circ_1, \circ_2, \ldots, \circ_{i+j} \} \) is a relational symbol, as for example one of \( \{ \equiv, \neq, =, \Rightarrow \}; \{ \circ_{i+1} B_{i+1} \cdots \circ_{i+j} B_{i+j} \} \) may be absent and if is present the validity of its presence is considered evident; each of \( \{ A_1, A_2, \ldots, A_i \} \) is replaced by symbol “\( \{ \} \)” when is considered evident the validity of the corresponding element of \( \{ \{ B_0 \circ_1 B_1 \}; \{ B_1 \circ_2 B_2 \}; \ldots; \{ B_{i-1} \circ_i B_i \}\} \).

A \( \{ A_h; h = \hat{h}, \hat{h} \} \) is a sequence, and then also a set, of \( \hat{h} - \hat{h} + 1 \) elements. Is implicit \( \{ h = \hat{h}, \hat{h} \} \Rightarrow \{ h; h = \hat{h}, \hat{h} \} \). A \( \{ A \parallel \mathcal{P} \} \) is the set of all the different specifications of \( A \) contextually possible when there is the condition \( \mathcal{P} \).

A bijection, i.e. a “one-to-one (injective) and onto (surjective)” correspondence, between two sets \( \underline{A} \) and \( \underline{B} \) of which \( \underline{A} \equiv \{ A_h; h = 1, \hat{h} \} \) and \( \underline{B} \equiv \{ B_k; k = 1, \hat{k} \} \), is a set of \( \hat{h} \) pairs indicated \( \underline{A} \leftrightarrow \underline{B} \) and defined by an \( \underline{A} \leftrightarrow \underline{B} \equiv \{ A_h, B_k(h); h = 1, \hat{h} \} \) of which \( \{ k_h; h = 1, \hat{h} \} \equiv \{ k = 1, \hat{k} \} \).

A \( \{ A_{hk}; h = 1, \hat{h}; k = 1, \hat{k} \} \) is the matrix that has \( \hat{h} \) rows, \( \hat{k} \) columns and \( A_{hk} \) as element of \( h \)-th row and \( k \)-th column. A \( \{ B_k; k = 1, \hat{k} \} \) is a column vector i.e. a matrix which has \( \hat{k} \) rows and one column.

It’s called \( \mathfrak{R}(G) \) the set of every different value that can have the quantity \( G \); and \( \mathfrak{R}(G) \), of which \( G \equiv \{ G_n; n = 1, \hat{n} \} \), the set of every different \( \hat{n} \)-tuple of values that can be respectively assumed by the quantities \( G \). A \( \hat{n} \)-tuple like this is also called point of \( \mathfrak{R}(G) \).

It is placed \( \underline{x} \equiv \{ x_n; n = 1, \hat{n} \} \). Every analytic function \( f(\underline{x}) \) is understood continuous in its definition domain. \( A f(\underline{x}) \in C^m(\mathfrak{R}) \) is equivalent to saying that \( f(\underline{x}) \) is of class \( C^m \) in \( \mathfrak{R} \) and indicates that every its mixed partial derivative of order less or equal to \( m \) is continuous at every \( \underline{x} \) of which \( \underline{x} \in \mathfrak{R} \). A value of \( f(\underline{x}) \) is said local when is understood in a particular point of \( \mathfrak{R}(\underline{x}) \).
It is implicit that a \( f(\bar{x}) \) expresses a variable \( f \) in the sense of \( f = f(\bar{x}) \). Hence

\[
\{f(\bar{x}), f(\bar{x})\} = \{\mathcal{R}(f) \supseteq \mathcal{R}(f(\bar{x})) \cup \mathcal{R}(f(\bar{x}))\}
\]

When they are not possible misunderstanding, as occurs as long as a variable is expressed by a only function, the function name may be replaced by that of the variable: for example one can write \( y' \) instead of \( y'(x) \). A \( f = f(\bar{x}) \), of which \( f = \{f_k; k = 1, \hat{k}\} \) or \( f(\bar{x}) = \{f_k(\bar{x}); k = 1, \hat{k}\} \), is equivalent to \( \{f_k = f_k(\bar{x}); k = 1, \hat{k}\} \).

2 Two models for approximate a function: the interpolating polynomial and the cubic spline.

Are considered the interpolating polynomial and the cubic spline between the models (for which reference is made in \([2],[3],[4],[5],[6],[7],[8]\)) for approximating a \( y(x) \) of which \( \mathcal{R}(x) \equiv [x_1, x_{\hat{p}}] \).

Both these models are interpolating i.e. they approximate \( y(x) \) on the basis of knowledge of interpolation points \( x \leftrightarrow y \) of which

\[
\begin{align*}
\hat{x} \leftrightarrow \hat{y} & \equiv \{x_p, y_p; p = 1, \hat{p}\} & x \equiv \{x_p; p = 1, \hat{p}\} & y \equiv \{y_p; p = 1, \hat{p}\} \\
y_p = y(x_p) & \{x_{p-1} < x_p; p = 2, \hat{p}\}
\end{align*}
\]

and they assume the \( y \) values at the respective \( x \).

2.1 The interpolating polynomial

The interpolating polynomial \( \mathcal{P}_1(x) \), which interpolates the \( x \leftrightarrow y \), is the polynomial, of degree at most \( \hat{p} - 1 \), expressed by

\[
\mathcal{P}_1(x) = \sum_{p=1}^{\hat{p}} \Gamma_p x^{p-1} = \sum_{p=1}^{\hat{p}} x^{p-1} \sum_{p=1}^{\hat{p}} x_{pp} y_p = \sum_{p=1}^{\hat{p}} y_p \lambda_{\text{ip}}(x) \tag{1}
\]

of which

\[
\begin{align*}
\{\Gamma_p; p = 1, \hat{p}\} & \equiv \Gamma = X^{-1} : y & x \equiv [x_p^{p-1}; p = 1, \hat{p}; p = 1, \hat{p}] \\
x^{-1} \equiv [x_{pp}; p = 1, \hat{p}; p = 1, \hat{p}] & \lambda_{\text{ip}}(x) \equiv \sum_{p=1}^{\hat{p}} x_{pp} x^{p-1} & \lambda'_{\text{ip}}(x) \equiv \sum_{p=1}^{\hat{p}} (p-1) x_{pp} x^{p-2}
\end{align*}
\]

The linear system \( x \cdot \Gamma = y \), that defines the coefficients \( \Gamma \) of \( \mathcal{P}_1(x) \), is equivalent to express \( \{\mathcal{P}_1(x_p) = y_p; p = 1, \hat{p}\} \) using the second member of (1). The (1) has at \( x = 0 \) a singular point which however, on the basis of (2.4.2.10) of \([1]\), it is understood eliminated by

\[
\mathcal{P}_1(0) = \lim_{x \to 0} (\Gamma_1 x^0 + \sum_{p=2}^{\hat{p}} \Gamma_p x^{p-1}) = \Gamma_1
\]
which follows from \( \lim_{x \to 0} x^0 = 1 \) (which also eliminates a possible indeterminate form \( 0^0 \) that could happen in \( X \)).

The \( X \), as well as its transpose \( X^T \), is a matrix type said of Vandermonde, of which

\[
\det X = \prod_{p=1}^{\hat{p}-1} \prod_{p=p+1}^{\hat{p}} x_p - x_p \neq 0
\]

that entail the existence of \( X^{-1} \) and so the existence and uniqueness of \( P_i(x) \).

The \( \delta_n \), of which \( \delta_n \equiv \{ \delta_{nn}; n = 1, \hat{n}; n = 1, \hat{n} \} \), \{ \delta_{nn} = 0; \forall n \neq n \} \) and \{ \delta_{nn} = 1; \forall n = n \}, is the identity matrix (also said unit matrix). For \( \Lambda \equiv [\Lambda_{mn}; m = 1, \hat{m}; n = 1, \hat{n}] \), \( |\Lambda| \equiv |A_m|; m = 1, \hat{m}; n = 1, \hat{n} \) defines \( |\Lambda| \) as the absolute value of \( \Lambda \). In the case \( \hat{m} = \hat{n} \), the numerical error of \( \Lambda \) inversion can be measured by the maximum value in the matrix \( |\Lambda| \cdot \Lambda^{-1} - \delta_n | \).

The \( X \) is typically ill-conditioned since the calculation of \( X^{-1} \) can induce that the maximum in \( |X \cdot X^{-1} - \delta_p| \) is important, even if \( \hat{p} \) is not large and the numbers are represented by a long sequence of digits. This inconvenience also relates to the calculation of \( \Gamma \) as solution of \( X \cdot \Gamma = y \).

The same \( P_i(x) \) expressed by (1) has also the Lagrange form

\[
P_i(x) = \sum_{p=1}^{\hat{p}} \frac{D_p(x)}{D_p(x_p)} y_p = D_i(x) \sum_{p=1}^{\hat{p}} \frac{y_p}{(x - x_p) D_p(x_p)}
\]

of which

\[
D_p(x) \equiv \prod_{p=1}^{\hat{p}} \delta_{pp} + (x - x_p)(1 - \delta_{pp}) \quad D_i(x) \equiv \prod_{p=1}^{\hat{p}} (x - x_p)
\]

and showing \( \lambda_{p,x} = D_p(x) / D_p(x_p) \),

\[
\lambda_{p,x} = D_p^{-1}(x_p) \sum_{a=1}^{\hat{p}} (1 - \delta_{ap}) \prod_{p=1}^{\hat{p}} \delta_{pa} + (1 - \delta_{pa})(\delta_{pp} + (1 - \delta_{pp})(x - x_p))
\]

The (2) is advantageously used in place of (1), since, by not requiring the calculation of \( X^{-1} \) nor the solution of \( X \cdot \Gamma = y \), implies fewer burdens and numerical errors.

Regardless of the particular form that expresses \( P_i(x) \), which affects only its numerical aspect, for its analytical error \( E_{p,x} \) is had

\[
E_{p,x} \equiv y(x) - P_i(x) = \frac{y^{(\hat{p})}(\xi_1(x))}{\hat{p}!} D(x)
\]

\[
E_{p}^{(k)}(x) \equiv y^{(k)}(x) - P_i^{(k)}(x) = \frac{y^{(\hat{p})}(\xi_2(x))}{(\hat{p} - k)!} \prod_{p=1}^{\hat{p} - k} x - \zeta_{kp}
\]
of which
\[ 1 \leq k \leq \hat{p} - 1 \]
\[ \{ \xi_i(x) \in (x_1, x_{\hat{p}}); i = 1, 2 \} \]
\[ \{ \zeta_{kp} \in (x_p, x_{p+k}); p = 1, \hat{p} - k \} \]
\[ \{ y(x) \in C^{\hat{p}}(\mathcal{R}_x) \} \Rightarrow \{(3), (4)\} \]

The (3) and (4) imply the respective
\[
|\varepsilon_{p1}(x)| = \frac{|y^{(p)}(\xi_1(x))|}{\hat{p}!} |\Delta(x)| \leq \frac{\Phi(y^{(p)})}{\hat{p}!} \prod_{p=1}^{\hat{p}}|x - x_p|
\]
\[
|\varepsilon_{p1}^{(k)}(x)| = \frac{|y^{(p)}(\xi_2(x))|}{\hat{p} - k)!} \prod_{p=1}^{\hat{p} - k}|x - \zeta_{kp}| \leq \frac{\Phi(y^{(p)})}{\hat{p} - k)!} \prod_{p=1}^{\hat{p} - k} \hat{\Delta}_p
\]
of which \( \Phi(y^{(p)}) \equiv \max\{|y^{(p)}(x)| \parallel x \in (x_1, x_{\hat{p}})\} \), \( \hat{\Delta}_p \equiv \max\{|x - x_p|, |x - x_{p+k}|\} \).

The \( \mathcal{P}_i(x) \) has as inconveniences oscillatory behavior and eventuality that this increases with \( \hat{p} \). Indeed, also sensibly expecting for most of cases that, with increasing the \( \hat{p} \) of \( x \) equidistant, the respective interpolating polynomials converge neatly to \( y(x) \), nevertheless this convergence is not always surely attainable, having however in this regard the improvement of replace the \( x \) equidistant with the Chebyshev points defined by
\[
\{ x_p = \frac{x_{\hat{p}} + x_1}{2} + \frac{x_{\hat{p}} - x_1}{2} \cos\left(\frac{\hat{p} - p}{\hat{p} - 1} \pi\right); p = 1, \hat{p}\}
\]

2.2 The cubic spline

The natural cubic spline, that interpolates the \( x \leftrightarrow y \), is the piecewise polynomial function made up by the \( \hat{p} - 1 \) polynomials \( \{ S_p(x) ; p = 1, \hat{p} - 1 \} \), in the sense of
\[ S(x) = S_p(x) \equiv \{ S_p(x) \parallel x \in \mathcal{J}_p\} \]
of which \( \mathcal{J}_p \equiv [x_p, x_{p+1}] \).

Each of \( \{ S_p(x) ; p = 1, \hat{p} - 1 \} \) is defined in the corresponding \( \mathcal{J}_p \) and has degree at most 3. The linear system of \( 4(\hat{p} - 1) \) equations
\[
\{ \begin{array}{l}
S_p(x_p) = y_p, S_p(x_{p+1}) = y_{p+1}; p = 1, \hat{p} - 1 \\
S'_{p-1}(x_p) = S'_p(x_p); p = 1, \hat{p} - 1 \\
S''_{p-1}(x_p) = S''_p(x_p); p = 2, \hat{p} - 1 \\
S'''_1(x_1) = S''_1(x_{\hat{p}}) = 0
\end{array} \]
\]
is solvable in just as many unknowns constituted by the coefficients of these \( \hat{p} - 1 \) polynomials.

The numerical burden of knowing the \( \{ S_p(x) ; p = 1, \hat{p} - 1 \} \) by solving this system with an algorithm of general applicability, as that of Gauss with the strategy of the
“maximum pivot” referred in [7] and [9], can be reduced by transforming such system as follows.  

Are established \( \{ u_p \equiv S_p''(x_p); p = 1, \hat{p} - 1 \} \) and \( v_{\hat{p}} \equiv S_{\hat{p}}''(x_{\hat{p}}) \), that allow both replace (8) with the understand \( \{ u_p \equiv S_{p-1}'(x_p) = S_p''(x_p); p = 2, \hat{p} - 1 \} \) and write (9) in the form \( u_1 = u_{\hat{p}} = 0 \).

The definition of \( S_p(x) \) as a polynomial of degree at most 3 implies that \( S_p''(x) \) is a polynomial of degree at most 1. This deduction, \( u_p = S_p''(x_p) \) and \( u_{p+1} = S_p''(x_{p+1}) \) entail

\[
\mathcal{E}(S_p''(x), \{ x_a, u_a; a = p, p + 1 \} \parallel \mathcal{P}_i(x), x \leftrightarrow y \parallel (2))
\]

that gives rise to

\[
S_p''(x) = \frac{u_{p+1}(x - x_p) - u_p(x - x_{p+1})}{\Delta_p}
\]

(10)
of which \( \Delta_p = x_{p+1} - x_p \) and that, on the basis of (5), show \( S(x) \in C^2(\mathfrak{R}_x) \).

From: \( S'_p(x) - v_p = \int S''_p(x) \, dx \), due to \( \{ F(x) = \int f(x) \, dx \equiv \{ F'(x) = f(x) \} \}; (10) \); follows

\[
S'_p(x) = \int S''_p(x) \, dx + v_p = \int \frac{u_{p+1}(x - x_p) - u_p(x - x_{p+1})}{\Delta_p} \, dx + v_p = \frac{u_{p+1}(x - x_p)^2 - u_p(x - x_{p+1})^2}{2\Delta_p} + v_p
\]

(11)

From: \( S_p(x) - w_p = \int S'_p(x) \, dx \); (11); follows

\[
S_p(x) = \int S'_p(x) \, dx + w_p = \int \frac{u_{p+1}(x - x_p)^2 - u_p(x - x_{p+1})^2}{2\Delta_p} \, dx + \int v_p \, dx + w_p = \frac{u_{p+1}(x - x_p)^3 - u_p(x - x_{p+1})^3}{6\Delta_p} + v_p(x - x_p) + w_p
\]

(12)

This deduction allows to write (6) as

\[
\{ w_p = y_p - \frac{\Delta^2 u_p}{6}, v_p = \frac{y_{p+1} - y_p}{\Delta_p} - \frac{u_{p+1} - u_p}{6} \Delta_p; p = 1, \hat{p} - 1 \}
\]

(13)
The (11) and expression of \( v_p \) in (13) allow to write (7) as

\[
\{ \frac{\Delta_p u_p}{6} + \frac{\Delta_p + \Delta_{p+2}}{3} u_{p+1} + \frac{\Delta_{p+1} u_{p+2}}{6} = y_p; p = 1, \hat{p} - 2 \}
\]
of which

\[
y_p = \frac{y_{p+2} - y_{p+1}}{\Delta_{p+1}} - \frac{y_{p+1} - y_p}{\Delta_p}
\]

and hence, using the matrix notation and introducing (9) in the form \( u_1 = u_{\hat{p}} = 0 \), as the linear system \( \mathbb{S} \cdot \underline{u} = \underline{y} \) of \( \hat{p} - 2 \) equations in the just as many unknowns \( \underline{u} \), defined by

\[
\underline{u} \equiv \{ u_p; p = 2, \hat{p} - 1 \} \quad \underline{y} \equiv \{ y_p; p = 1, \hat{p} - 2 \} \quad \mathbb{S} \equiv [s_{pp}; p = 1, \hat{p} - 2; p = 1, \hat{p} - 2]
\]
with the elements of $\Xi$ all null except

$$S_{11} = \frac{\Delta_1 + \Delta_2}{3}, \quad S_{12} = \frac{\Delta_2}{6}, \quad S_{p-2,p-2} = \frac{\Delta_{p-2}}{3}, \quad S_{p-2,p-3} = \frac{\Delta_{p-2}}{6}, \quad S_{\hat{p},p} = \frac{\Delta_{p} + \Delta_{p+1}}{3}, \quad S_{p,p+1} = \frac{\Delta_{p+1}}{6}; \quad p = \hat{p} - 3$$

and being such system equivalent to $U = \Xi^{-1} \cdot \mathcal{Y}$ of which $\Xi^{-1} \equiv [S_{pp}; p = 1, \hat{p} - 2; p = 1, \hat{p} - 2]$ and hence to

$$\{U_{p+1} = \sum_{p=1}^{\hat{p}-2} S_{pp} y_p; p = 1, \hat{p} - 2\}$$

The system constituted by (6), (7), (8) and (9) was transformed in that constituted by (13), $\Xi = \Xi^{-1} : \mathcal{Y}$ and $U_1 = U_{\hat{p}} = 0$. Therefore the introduction of these into (12) and (11) makes known the expressions of $\{s_p(x), s'_p(x); p = 1, \hat{p} - 1\}$, having in particular

$$\{s'(x_p) = s'_p(x_p) = \sum_{p=1}^{\hat{p}} \lambda_{pp} y_p; \forall p < \hat{p}\} \quad \{s'(x_{\hat{p}}) = s'_{\hat{p}}(x_{\hat{p}}) = \sum_{p=1}^{\hat{p}} \vartheta_{p} y_{p}\}$$

of which

$$\lambda_{pp} = \frac{\delta_{p,p+1} \delta_{p1} - \delta_{pp} \delta_{pp}}{\Delta_p} - \left( \frac{\delta_{p1} \delta_{pp} \kappa_{pp}}{3} + \frac{\delta_{p+1,p} \kappa_{p+1,p}}{6} \right) \Delta_p$$

$$\vartheta_{p} = \frac{\delta_{p} \delta_{p,p-1} - \delta_{pp} \delta_{pp-1}}{\Delta_{p-1}} + \frac{\delta_{p-1} \delta_{pp} \kappa_{p-1,p}}{6} \Delta_{p-1}$$

$$\kappa_{pp} = \frac{\delta_{p,p,p} \delta_{pp} \delta_{pp-1,p}}{\Delta_p} - \delta_{pp} \delta_{p1} \delta_{p-1} \left( \Delta_{p-1} + \Delta_{p} \right) \Delta_{p-1} + \frac{\delta_{p+1} \delta_{p2} \delta_{pp-1,p-2}}{\Delta_{p-1}}$$

where is understood $\delta_{ab} = 1 - \delta_{ab}$ whence $\{\delta_{ab} = 1; \forall a \neq b\}, \{\delta_{ab} = 0; \forall a = b\}$, and is considered null every addend where it appears at least a factor null.

The complete cubic spline $S_c(x)$ differs from $S(x)$ only for the substitution of (9) with the assignment of known values to $s'_p(x_1)$ and $s'_p(x_{\hat{p}})$. The periodic cubic spline $S_p(x)$ differs from $S(x)$ only for the substitution of (9) with $s'_p(x_1) = s'_p(x_{\hat{p}})$ and $s''_p(x_1) = s''_p(x_{\hat{p}})$ when $y(x_1) = y(x_{\hat{p}})$.

For $S_c(x)$ we have (in [5]) the

$$|E_{sc}(x)| = |y(x) - S_c(x)| \leq \frac{7}{8} \Phi(y^{(4)}) \frac{\hat{\Delta}^5}{\Delta}$$

$$|E'_{sc}(x)| = |y'(x) - S'_c(x)| \leq \frac{7}{4} \Phi(y^{(4)}) \frac{\hat{\Delta}^4}{\Delta}$$

of which

$$\Phi(y^{(p)}) = \max\{|y^{(p)}(x)| \parallel x \in \mathcal{R}_x\} \quad \hat{\Delta} \equiv \max\{\Delta_p; p = 1, \hat{p} - 1\}$$

$$\hat{\Delta} \equiv \min\{\Delta_p; p = 1, \hat{p} - 1\} \quad \{y(x) \in C^4(\mathcal{R}_x) \Rightarrow (15)\}$$

8
Other bounds for the errors of the cubic spline are in [2], [3], [4], [8].

With the increase of \( \hat{p} \), each subsequent \( S_c(x) \) and its derivatives up to second order converges to \( y(x) \) and its corresponding derivative, with the only condition that \( \hat{\Delta}/\hat{\Delta} \) remains limited, and having in the case of \( \hat{x} \) equidistant the most rapid convergence indicated by

\[
|\mathcal{E}_{sc}(x)| \leq \frac{7}{8} \hat{\Delta}^4 \hat{\Phi}(\gamma^{(4)})
\]

In the limit as \( \hat{p} \to \infty \), i.e. as \( \hat{p} \) approaches \( \infty \), consistent with the said condition, the (15) are valid also for \( S(x) \) inasmuch this coincides with \( S_c(x) \).

The inconveniences of \( P_i(x) \), said in section 2.1, are resolved in an excellent way by \( S(x) \), because the oscillations of this are minimal among those of all the different functions of class \( C^2 \) in \( \mathfrak{R}_x \) which interpolate the \( \hat{x} \Leftrightarrow \gamma \), and for the said properties of convergence to the exact value with the increase of \( \hat{p} \).

### 2.3 New bounds for the errors of a cubic spline

It is called \( \hat{S}(x) \) a cubic spline which differs from \( S(x) \) only for the replacement of (9) with another two equations, thus having \( \mathcal{A}(\hat{S} \parallel \hat{S}) \), \( \mathcal{A}(S_c \parallel \hat{S}) \) and \( \mathcal{A}(S_p \parallel \hat{S}) \). Coherently with this, is understood the possibility of replacing \( S \) with \( \hat{S}(x) \) when treating properties independent from (9).

From: \( P_{g(\hat{x})} \) of which \( P_{g(\hat{x})} \equiv \{ g(\hat{x})/\hat{y}(\hat{x}) \} = \omega_{g(\hat{x})}; \forall \hat{x} \in \mathfrak{R}_{\hat{x}} \} \) with \( \omega_{g(\hat{x})} \) constant; first mean value theorem for integration said in section 2.4.4 of [1]; follows

\[
\int_{\mathfrak{R}_{\hat{x}}} f(\hat{x}) g(\hat{x}) \, d\hat{x} = \omega_{g(\hat{x})} \int_{\mathfrak{R}_{\hat{x}}} f(\hat{x}) |g(\hat{x})| \, d\hat{x} = f(\hat{x}) \int_{\mathfrak{R}_{\hat{x}}} g(\hat{x}) \, d\hat{x}
\]

of which \( P_{g(\hat{x})} \Rightarrow (16), \hat{x} \in \mathfrak{R}_{\hat{x}} \), and that gives rise to \( \int_{\mathfrak{R}_{\hat{x}}} f(\hat{x}) \, d\hat{x} = \int_{\mathfrak{R}_{\hat{x}}} g(\hat{x}) \, d\hat{x} \) if \( g(\hat{x}) = 1 \).

The (10) and (5) entail

\[
\begin{align*}
x \in (x_p, x_{p+1}) & \Rightarrow \{ \hat{S}^{(3)}(x) = \frac{U_{p+1} - U_p}{\Delta_p} \} \\
\hat{S}^{(3)}(x_1) & = \frac{U_2 - U_1}{\Delta_1} \\
\hat{S}^{(3)}(x_{\hat{p}}) & = \frac{U_{\hat{p}} - U_{\hat{p}-1}}{\Delta_{\hat{p}-1}}
\end{align*}
\]

Are considered as implicit

\[
\begin{align*}
y(x) & \in C^4(\mathfrak{R}_x) \\
\mathcal{E}^{(o)}_p & \equiv \mathcal{E}^{(o)}(x_p) \\
\mathcal{E}^{(o)}(x) & \equiv \hat{S}^{(o)}(x) - y^{(o)}(x) \\
\mathfrak{R}(t_p) & \equiv (x_p, x_{p+1}) \\
i_p(t) & \equiv t - x_p \\
i_{p+1}(t) & \equiv t - x_{p+1} \\
\omega(G) & \equiv G/|G|
\end{align*}
\]

The

\[
\begin{align*}
\{ \mathcal{A}(x_p, x_{p+1} \parallel x \parallel (2)), t \in \mathcal{J}_p \} & \Rightarrow \{ \mathcal{P}_i(t) = \frac{i_p(t)y_{p+1} - i_{p+1}(t)y_p}{\Delta_p} \} \\
\{ \mathcal{A}(x_p, x_{p+1}, \mathcal{E}^{(o)}_{p+1} \parallel x \parallel (3)), t \in \mathcal{J}_p \} & \Rightarrow \{ \mathcal{E}^{(o)}(t) - \mathcal{P}_i(t) = \frac{i_p(t)i_{p+1}(t)\mathcal{E}^{(4)}(\xi_{\Delta}(t))}{2} \}
\end{align*}
\]
of which \( \{ t \in \mathcal{I}_p \} \Rightarrow \{ \xi(t) \in \mathcal{R}_t \} \); and \( \{ t \in \mathcal{R}_t \} \Rightarrow \{ \mathcal{E}^{(4)}(t) = -y^{(4)}(t) \} \) due to (17), entail \( A_p(t_p) = 0 \) of which

\[
A_p(t) \equiv \tilde{A}_p(t) + \frac{i_p(t) \Delta y^{(4)}(\xi(t))}{2} \quad \tilde{A}_p(t) \equiv \Delta p \mathcal{E}''(t) + \frac{i_p(t) \mathcal{E}'' - i_p(t) \mathcal{E}'''}{2}
\]

From: these, distributive property of integration and

\[
\int_{\mathcal{R}_t} f(x) \, dx = \lim_{\mathcal{R} \to \mathcal{R}_t} \int_{\mathcal{R}} f(x) \, dx;
\]

(16), and constancy of \( \omega(i_p(t) \tilde{i}_p(t)) \) if \( t \in \mathcal{I}_p \); \( i_p(t) \tilde{i}_p(t) = \frac{i^2}{p(t)} - \Delta p \tilde{i}_p(t) \) due to \( \Delta p \equiv \tilde{i}_p(t) - i_p(t) \);

\[
\lim_{a \to x_p} \int_a^{t_p} A_p(t) \, dt = 0
\]
due to \( A_p(t_p) = 0 \) and \( \mathcal{R}_t \equiv (x_p, x_{p+1}) \); follows

\[
\lim_{a \to x_p} \int_a^{t_p} A_p(t) \, dt = \int_{x_p}^{t_p} \tilde{A}_p(t) \, dt + \frac{\Delta p}{2} \int_{x_p}^{t_p} i_p(t) \tilde{i}_p(t)y^{(4)}(\xi(t)) \, dt = \int_{x_p}^{t_p} \tilde{A}_p(t) \, dt + \frac{\Delta p}{2} y^{(4)}(\xi(a(t))) \int_{x_p}^{t_p} i_p(t) \tilde{i}_p(t) \, dt = B_p(t_p) = 0 \hspace{1cm} (18)
\]
of which \( \xi(a(t_p)) \in \mathcal{R}_t, \) and

\[
B_p(t) \equiv \Delta p\mathcal{E}'(t) - \Delta p\mathcal{E}' + \frac{i^2}{p(t)} - \Delta p^2 \mathcal{E}'' - \frac{i^2}{p(t)} \mathcal{E}'' - \frac{(\Delta p t^3(t) - \Delta p^2 t^2(t))}{6} \Delta p y^{(4)}(\xi(a(t)))
\]

From: \( \mathcal{E}_p = 0; \) (16), constancy of \( \omega(i_p(t)/6 - \Delta_p/4) \) if \( t \in \mathcal{I}_p \); \( \int_{x_p}^{t_p} B_p(t) \, dt = 0 \) due to \( B_p(t_p) = 0 \) affirmed by (18) and to \( \mathcal{R}_t \equiv (x_p, x_{p+1}) \); follows

\[
\int_{x_p}^{t_p} B_p(t) \, dt = \Delta p\mathcal{E}(t_p) - \Delta p \tilde{i}_p(t_p)\mathcal{E}' + \frac{i^3}{p(t)} - \Delta p - 3\Delta p \tilde{i}_p(t_p) \mathcal{E}'' - \frac{i^3}{p(t)} \mathcal{E}_{p+1}' + \Delta p \int_{x_p}^{t_p} (i_p(t)/6 - \Delta_p/4) i^2_p(t) y^{(4)}(\xi(a(t))) \, dt = C_p(t_p) = 0 \hspace{1cm} (19)
\]
of which, being \( \xi_c(t_p) \in \mathcal{R}_t, \) is had

\[
C_p(t) \equiv \Delta p\mathcal{E}(t) - \Delta p \tilde{i}_p(t)\mathcal{E}' + \frac{i^3}{p(t)} - 3\Delta p t^2(t) \mathcal{E}'' - \frac{i^3}{p(t)} \mathcal{E}_{p+1} + \Delta p (\frac{i^4}{p(t)} - \frac{\Delta p t^3(t)}{12}) y^{(4)}(\xi_c(t))
\]
By implementing for \( \{x_{p+1}, t_p\} \) a process similar to that for \( \{x_p, t_p\} \) has led to (18) and (19) starting from \( A_p(t_p) = 0 \), are obtained \( D_p(t_p) = 0 \) and \( E_p(t_p) = 0 \) of which

\[
D_p(t) \equiv \Delta_p E'(t) - \Delta_p E'_{p+1} + \frac{i_p^2(t)}{2} \mathcal{E}''_{p+1} + \frac{\Delta_p^3 - i_p^2(t)}{2} \mathcal{E}''_{p+1} + \frac{\Delta_p^3}{6} \left( \frac{i_p^3(t)}{6} - \frac{\Delta_p^3}{12} - \frac{i_p^2(t) \Delta_p}{4} \right) \Delta_p y^{(4)}(\xi_d(t))
\]

\[
E_p(t) \equiv \Delta_p E(t) - \Delta_p E'_{p+1} + \frac{i_p^3(t)}{6} \mathcal{E}''_{p+1} + \frac{3i_p(t) \Delta_p^2 + \Delta_p^3 - i_p^3(t)}{6} \mathcal{E}''_{p+1} + \frac{i_p^4(t) - \Delta_p^4}{24} + \frac{\Delta_p^2 i_p(t) - \Delta_p^4 - \Delta_p^3 \Delta_p y^{(4)}(\xi_e(t))}{12}
\]

where \( \xi_d(t_p) \in \mathcal{R}_{t_p} \) and \( \xi_e(t_p) \in \mathcal{R}_{t_p} \).

The limit of \( B_p(t_p) = 0 \) as \( t_p \to x_{p+1} \) gives rise, intending \( \zeta_\lambda \in \mathcal{R}_{t_p} \), to

\[
\mathcal{E}'_p - \mathcal{E}'_{p+1} + \frac{\mathcal{E}''_p + \mathcal{E}''_{p+1}}{2} \Delta_p + \frac{\Delta_p^3}{12} y^{(4)}(\zeta_\lambda) = 0
\]

(20)

The limit of \( C_p(t_p) = 0 \) as \( t_p \to x_p \) gives rise, considering also \( \mathcal{E}_{p+1} = 0 \) and intending \( \zeta_\eta \in \mathcal{R}_{t_p} \), to

\[
\mathcal{E}'_p + \frac{\Delta_p}{3} \mathcal{E}''_p + \frac{\Delta_p}{6} \mathcal{E}''_{p+1} + \frac{\Delta_p^3}{24} y^{(4)}(\zeta_\eta) = 0
\]

(21)

The limit of \( E_p(t_p) = 0 \) as \( t_p \to x_p \) gives rise, intending \( \zeta_\zeta \in \mathcal{R}_{t_p} \), to

\[
\mathcal{E}'_{p+1} + \frac{\Delta_p}{6} \mathcal{E}''_p - \frac{\Delta_p}{3} \mathcal{E}''_{p+1} - \frac{\Delta_p^3}{24} y^{(4)}(\zeta_\zeta) = 0
\]

(22)

The Rolle’s theorem ([10],[11]) asserts

\[
\{ f(x) \in C^0[a,b], f(x) \in C^1(a,b), f(a) = f(b) \} \Rightarrow \exists \{ f'(x) = 0 \mid x \in (a,b) \}
\]

(23)

Is placed

\[
F(x) \equiv \mathcal{E}''(x) - \frac{i_p(x) \mathcal{E}''_{p+1} - i_p(x) \mathcal{E}''}{\Delta_p}
\]

This, \( \hat{S}(x) \in C^2(\mathcal{R}_x) \) and \( \hat{S}(x) \in C^3(\mathcal{R}_x - \{x_p; p = 2, \hat{p} - 1\}) \) (due to (17)) entail

\[
F(x_p) = F(x_{p+1}) = 0 \quad F(x) \in C^0(\mathcal{J}_p) \quad F(x) \in C^1(\mathcal{X}_p, \mathcal{X}_{p+1})
\]

This deduction and \( \mathcal{A}(F(x), x_p, x_{p+1}/f(x), a, b/(23)) \) entail, intending \( \zeta_\zeta \in (x_p, x_{p+1}) \),

\[
\mathcal{E}'_p - \mathcal{E}'_{p+1} + \Delta_p \mathcal{E}^{(3)}(\zeta_\zeta) = 0
\]

(24)
Are placed the
\[ \{p_A, p_B\} \subseteq \{p = 1, p = \hat{p} - 1\} \quad p_B > p_A \quad \square \cup \{J_p; p = 1, p = \hat{p} - 1\} \quad \mathbb{L} \subseteq \{L_p; p = 1, p = \hat{p} - 1\} \]
\[ \{J_p; \forall p \in \{p = p_A, p_B\}\} \quad \{J_p = \{1 \forall 2\}; \forall p \in \{p = p_A, p_B\}\} \]
\[ \{\mathbf{u}, \mathbf{v}\} \subseteq \{p = p_A, p_B\} \quad \{L_p = \{\forall p \neq \mathbf{u}, \forall p \neq \mathbf{v}\}\} \quad \mu = \mathbb{L} = 1 \quad (25) \]

The writing of \( A_p(t_p) = 0, B_p(t_p) = 0, C_p(t_p) = 0, D_p(t_p) = 0, E_p(t_p) = 0, (20), (21), (22) \) and (24) for each of \( \{p = 1, \hat{p} - 1\}\), subordinated to (25) in the sense that these specify as follows those that are replaced by the same number of expressions all equal to \( 0 = 0 \), gives rise to the homogeneous linear system \( \overrightarrow{M} \cdot \mathbf{\nu}' = 0 \beta \) (of which \( 0(\beta) \equiv [0; \iota = 1, \mathbb{A}] \)) defined by \( \mathbb{M} \equiv [\mathcal{M}_{\mathbb{A}}; r = 1, \mathbb{M}; c = 1, \mathbb{n}], \mathbb{M} \equiv \{p, q = 1, \hat{p} - 1; q = 1, 8\} \) with \( \mathcal{Q}_{\mathcal{Q}_p} \) a q-th value of \( y^{(q)}(x) \) in \( (x_p, x_{p+1}) \), and by the nullity of all the elements of \( \overrightarrow{M} \) that are not defined by

\[ \mathcal{M}(H(p, 1), K(2, 0) + p) = \Delta_p \delta_{1J_p} \quad \mathcal{M}(H(p, 1), K(3, 1) + p) = \hat{I}_p(t_p) \delta_{1J_p} \]
\[ \mathcal{M}(H(p, 1), K(3, 1) + p + 1) = -\hat{I}_p(t_p) \delta_{1J_p} \]
\[ \mathcal{M}(H(p, 1), K(4, 2) + p) = -\Delta_p \hat{I}_p(t_p) \hat{I}_p(t_p) \delta_{1J_p}/2 \]
\[ \mathcal{M}(H(p, 2), K(1, 0) + p) = \Delta_p \delta_{1J_p} \quad \mathcal{M}(H(p, 2), K(3, 0) + p) = -\Delta_p \delta_{1J_p} \]
\[ \mathcal{M}(H(p, 2), K(3, 1) + p) = \left(\hat{I}^2_p(t_p) - \Delta^2_p\right) \delta_{1J_p}/2 \]
\[ \mathcal{M}(H(p, 2), K(3, 1) + p + 1) = -\hat{I}^2_p(t_p) \delta_{1J_p}/2 \]
\[ \mathcal{M}(H(p, 2), K(5, 2) + p) = \Delta_p \hat{I}_p(t_p) \left(\Delta_p/4 - \hat{I}_p(t_p)/6\right) \delta_{1J_p} \]
\[ \mathcal{M}(H(p, 3), K(3, 0) + p) = \Delta_p \delta_{1J_p} \quad \mathcal{M}(H(p, 3), K(3, 0) + p) = -\Delta_p \delta_{1J_p} \hat{I}_p(t_p) \]
\[ \mathcal{M}(H(p, 3), K(3, 1) + p) = \left(\hat{I}^3_p(t_p) - 3\Delta_p \hat{I}^2_p(t_p)\right) \delta_{1J_p}/6 \]
\[ \mathcal{M}(H(p, 3), K(3, 1) + p + 1) = -\hat{I}^3_p(t_p) \delta_{1J_p}/6 \]
\[ \mathcal{M}(H(p, 3), K(6, 2) + p) = \Delta_p \hat{I}_p(t_p) \left(\Delta_p/12 - \hat{I}_p(t_p)/24\right) \delta_{1J_p} \]
\[ \mathcal{M}(H(p, 4), K(1, 0) + p) = \Delta_p \delta_{1J_p} \quad \mathcal{M}(H(p, 4), K(3, 0) + p + 1) = -\Delta_p \delta_{1J_p} \]
\[ \mathcal{M}(H(p, 4), K(3, 1) + p) = \left(\hat{I}^2_p(t_p) - 3\Delta_p \hat{I}^2_p(t_p)\right) \delta_{1J_p}/6 \]
\[ \mathcal{M}(H(p, 4), K(3, 1) + p + 1) = -\hat{I}^2_p(t_p) \delta_{1J_p}/2 \]
\[ \mathcal{M}(H(p, 4), K(7, 2) + p) = \Delta_p \left(\Delta_p \hat{I}_p(t_p)/4 - \Delta^3_p/12 - \hat{I}^3_p(t_p)/6\right) \delta_{1J_p} \]
\[ M(\mathcal{H}(p, 5), p) = \Delta p \delta_{1L_p} \quad M(\mathcal{H}(p, 5), \mathcal{K}(3, 0) + p + 1) = -\Delta p \delta_{1L_p} \]
\[ M(\mathcal{H}(p, 5), \mathcal{K}(3, 1) + p) = \frac{\Delta^3 p}{6} \delta_{1L_p} \]
\[ M(\mathcal{H}(p, 5), \mathcal{K}(3, 1) + p + 1) = \left( 3\Delta^2 p \delta_{1p}(t) + \Delta^3 p- \Delta^3 p(t) \right) \delta_{1L_p} / 6 \]
\[ M(\mathcal{H}(p, 5), \mathcal{K}(8, 2) + p) = \Delta p \left( \left( \Delta^4 p - \Delta^3 p(t) \right) / 24 - \Delta p \left( \Delta^2 p \delta_{1p}(t) - \Delta^3 p(t) + \Delta^3 p \right) / 12 \right) \delta_{1L_p} \]

of which \( p \in \{ p = 1, \hat{p} - 1 \} \), \( \mathcal{H}(a, b) \equiv 9(a - 1) + b \) and \( \mathcal{K}(a, b) \equiv a(\hat{p} - 1) + b \hat{p} \).

From \( M \) is obtained a matrix \( N \) with the following steps:

1. are posed the \( N \equiv [N_{r; c}, r = 1, \hat{m}; c = 1, \hat{n}] \equiv M \{ r_m; m = 1, \hat{m} \} = \{ m = 1, \hat{m} \}, \{ c_n; n = 1, \hat{n} \} = \{ n = 1, \hat{n} \}, R=1; \)
2. if \( R = \mathcal{K}(3, 2) + 1 \) is executed step 8;
3. is posed \( P = \max \{ |N(r_m, c_n)|; m = R, \hat{m}; n = R, \mathcal{K}(3, 2) \} \);
4. if \( P = 0 \) is executed step 8;
5. is posed \( \{ r, c \} = \{ m, n \} \parallel P = |N(r_m, c_n)| \}, \) are exchanged the values between \( r_r \) and \( r_t \) and between \( c_n \) and \( c_c \);
6. are replaced the \( \{ N(r_m, c_n); m = R + 1, \hat{m}; n = R + 1, \hat{n} \} \) with the respective \( \{ N(r_m, c_n) - N(r_m, c_n)N(r_{r_n}, c_n)/N(r_{r_n}, c_n); m = R + 1, \hat{m}; n = R, \hat{n} \} \);
7. \( R \) is incremented by 1 and it is returned to step 2;
8. \( R \) is decreased by 1 and are replaced the \( \{ N(r_m, c_n); n = m, \hat{n}; m = 1, R \} \) with the respective \( \{ N(r_m, c_n)/N(r_m, c_n); n = m, \hat{n}; m = 1, R \} \);

13
9. are performed the iterations indicated by \( k = 2, R \) and at the \( k \)-th are replaced the \( \{N(r_m, c_n); m = 1, k - 1; n = k, n\} \) with the respective \( \{N(r_m, c_n) - N(r_k, c_n)N(r_m, c_k); m = 1, k - 1; n = k, n\} \).

The \( \mathbb{N} \) that is had after this execution verifies still the \( \mathbb{N} \cdot \mathcal{V} = 0_m \) as when it was placed the \( \mathbb{N} ≡ \mathcal{M} \), with the advantage that each of the rows, indicated by \( \{r_k; k = 1, R\} \), of this additional homogeneous linear system gives rise to a corresponding

\[
\mathcal{V}_c = -\sum_{q=1}^{9} \sum_{p=1}^{8} N(r_k, K(q + 3, 2) + p)Q_{qp} - N(r_k, K(3, 2) + u)E^{(3)}(\zeta_u) - N(r_k, K(3, 2) + v)E^{(3)}(\zeta_v)
\]

(26)
of which

\[
\mathcal{V}_c \in \{ \{\mathcal{E}(t_p); p = p_A, p_B\}, \{\mathcal{E}'(t_p); p = p_A, p_B\}, \{\mathcal{E}''(t_p); p = p_A, p_B\}, \{\mathcal{E}'_p; p = p_A, p_B + 1\}, \{\mathcal{E}''_p; p = p_A, p_B + 1\} \}
\]

This, (24) and \( p \in \{ p = p_A, p_B + 1 \} \) entail

\[
\mathcal{E}'_p = \alpha_p + \beta_{up}(\mathcal{E}''_{u_1} - \mathcal{E}_u') + \beta_{vp}(\mathcal{E}''_{v_1} - \mathcal{E}_v')
\]

of which

\[
\alpha_p = -\sum_{q=1}^{9} \sum_{p=1}^{8} N(r_{ap}, K(q + 3, 2) + p)Q_{qp}
\]

\[
\beta_{up} \equiv \frac{-N(r_{ap}, K(3, 2) + u)}{\Delta_u}
\]

where \( a_p \equiv \{ k \mid c_k = K(3, 1) + p \} \).

This implies

\[
\mathcal{E}_{p+1}'' - \mathcal{E}_p'' = \alpha_p + \beta_{up}(\mathcal{E}''_{u_1} - \mathcal{E}_u'') + \beta_{vp}(\mathcal{E}''_{v_1} - \mathcal{E}_v'')
\]

of which \( \alpha_p \equiv \alpha_{p+1} - \alpha_p, \beta_{up} \equiv \beta_{u,p+1} - \beta_{up} \), and that, by introducing \( p \equiv u \) or \( p \equiv v \), gives rise, intending \( \beta_{uu} \equiv 1 - \beta_{uu} \), to the respective

\[
\beta_{uu}(\mathcal{E}''_{u_1} - \mathcal{E}_u'') - \beta_{uv}(\mathcal{E}''_{v_1} - \mathcal{E}_v'') = \alpha_u
\]

\[
\beta_{uv}(\mathcal{E}''_{u_1} - \mathcal{E}_u'') - \beta_{vv}(\mathcal{E}''_{v_1} - \mathcal{E}_v'') = -\alpha_v
\]

By solving with the Cramer’s method ([10]) the system in the unknowns \( \mathcal{E}''_{u_1} - \mathcal{E}_u'' \)
and \( \mathcal{E}''_{v_1} - \mathcal{E}_v'' \) constituted by these two equations, is had

\[
\mathcal{E}''_{u_1} - \mathcal{E}_u'' = \frac{\hat{\beta}_{uv}\alpha_u + \hat{\beta}_{vv}\alpha_u}{\theta_{uv}}
\]

\[
\mathcal{E}''_{v_1} - \mathcal{E}_v'' = \frac{\hat{\beta}_{uv}\alpha_u + \hat{\beta}_{uu}\alpha_v}{\theta_{uv}}
\]
of which \( \tilde{\theta}_{uv} = \hat{\theta}_{uv} - \hat{\theta}_{wu} \).

This and the fact that occur \( \hat{\theta}_{uu} = \hat{\theta}_{vv} = -1 \) and \( \hat{\theta}_{uv} = 0 \), imply \( \mathcal{E}_{u+1}'' - \mathcal{E}_v'' = \tilde{\alpha}_u/2 \)
and \( \mathcal{E}_{v+1}'' - \mathcal{E}_v'' = \tilde{\alpha}_v/2 \). These and (24) entail \( \mathcal{E}^{(3)}(\zeta_u) = \tilde{\alpha}_u/(2\Delta_u) \) and \( \mathcal{E}^{(3)}(\zeta_v) = \tilde{\alpha}_v/(2\Delta_v) \). Introducing these and \( \tilde{\alpha}_p = \tilde{\alpha}(\mathcal{Q}) \) in (26), is had \( \mathcal{Q}_{c_k} = - \sum_{q=1}^{8} \sum_{p=1}^{\hat{\rho}-1} \tilde{\mathcal{N}}_{uvqp} Q_{qp} \)
of which

\[
\tilde{\mathcal{N}}_{uvqp} \equiv N(r_k, K(q + 3, 2) + p) + \\
\frac{2\Delta_u}{N(r_{a_r}, K(q + 3, 2) + p) - N(r_{a_{r+1}}, K(q + 3, 2) + p)} N(r_k, K(3, 2) + u) + \\
\frac{2\Delta_v}{N(r_{a_r}, K(q + 3, 2) + p) - N(r_{a_{r+1}}, K(q + 3, 2) + p)} N(r_k, K(3, 2) + v)
\]

and that gives rise to

\[
|\mathcal{Q}_{c_k}| = \left| \sum_{q=1}^{8} \sum_{p=1}^{\hat{\rho}-1} \tilde{\mathcal{N}}_{uvqp} Q_{qp} \right| \leq \\
\sum_{q=1}^{8} \sum_{p=1}^{\hat{\rho}-1} |\tilde{\mathcal{N}}_{uvqp} Q_{qp}| \leq \sum_{p=1}^{\hat{\rho}-1} \Phi_p(\mathcal{Y}^{(4)}) \sum_{q=1}^{8} |\tilde{\mathcal{N}}_{uvqp}| \leq \Phi(\mathcal{Y}^{(4)}) \sum_{p=1}^{\hat{\rho}-1} \sum_{q=1}^{8} |\tilde{\mathcal{N}}_{uvqp}|
\]

(27)
of which

\[
\Phi_p(\mathcal{Y}^{(4)}) = \max\{|\mathcal{Y}^{(4)}(x)| / x \in \mathfrak{R}_{t_p}\} \quad \Phi(\mathcal{Y}^{(4)}) = \max\{|\mathcal{Y}^{(4)}(x)| / x \in (x_1, x_p)\}
\]

Each different choice of the \( p_A, p_B, \underline{q}, \underline{u}, \underline{v} \) and \( \{t_p; p = 1, \hat{\rho} - 1\} \) gives rise to a correspondingly different (27) which can possibly enable a further lower upper bound for some of the \( |\mathcal{Q}_{c_k}|; c = 1, K(3, 2) \). Hence for \( \{\mathcal{E}_p'; p = 1, \hat{\rho}\} \) is in particular possible to proceed as follows. Is placed \( \hat{K}_p \equiv \infty; p = 1, \hat{\rho} \) and are chosen two natural numbers \( \hat{N}_{ir} \) and \( \hat{N}_{in} \) of which \( \hat{N}_{in} \geq 2 \). Are carried the iterations indicated by \( \{N_{ir} = 2, \hat{N}_{ir}\} \). For every \( \hat{N}_{ir} \), if \( \hat{\rho} - \hat{N}_{ir} \geq 1 \), are carried the iterations indicated by \( \{p_A = 1, \hat{\rho} - \hat{N}_{ir}\} \). In the \( p_A \)-th of these iterations, is placed \( p_B = p_A + \hat{N}_{ir} - 1 \) and are carried the iterations indicated by \( \{t_1 = 1, \hat{N}_{ir}\} \) of which \( N_{ir} > N_{ir}; \forall \{p_A = 1\} \forall \{p_B = \hat{\rho} - \hat{N}_{ir}\} \) and \( N_{ir} = N_{ir} \forall p_A \in \{p = 2, \hat{\rho} - \hat{N}_{ir} - 1\} \). For every \( t_1 \), are chosen \( \{t_p, t_p; p = p_A, p_B\} \) and \( \{u, v\} \) in conformity with (25), are calculated the corresponding \( \{K_p; p = p_A, p_B + 1\} \)
using \( \sum_{p=1}^{\hat{\rho}-1} \sum_{q=1}^{8} |\tilde{\mathcal{N}}_{uvqp}| \) and \( k = \{k || c_k = K(3, 2) + p\} \), and for every \( \{p = p_A, p_B + 1\} \) is replaced \( K_p \) with \( K_p \) if \( K_p < K_p \). After these steps is had

\[
\{\mathcal{E}_p' \leq K_p \Phi(\mathcal{Y}^{(4)}); p = 1, \hat{\rho}\}
\]

(28)
which generally improves with the increase of \( \hat{N}_{ir} \) and, for the evident greater influence on \( \mathcal{E}_p' \) of those of the \( \{\mathcal{J}_p; p = 1, \hat{\rho} - 1\} \) that are closer to it, even more with the increase of \( \hat{N}_{ir} \).
3 The maximum absolute value of a derivative defined on a curve of the multidimensional Euclidean space

A vector, also called free vector, is a straight line segment, defined by a direction (it can lie on whichever straight line identifiable as an element of a corresponding infinite set of parallel straight lines), by a sense (its extreme points are distinct as initial and terminal) and by a length (or magnitude) which is its measure. A vector applied at a point is a vector which has such point as initial. A versor, also called unit vector, is a vector which has unitary magnitude.

For a vector \( \overrightarrow{x} \), \( |\overrightarrow{x}| \) (of which \( |\overrightarrow{x}| \equiv x \geq 0 \)) and \( \overrightarrow{v}(x) \) indicate respectively magnitude and versor of \( \overrightarrow{x} \), being \( \overrightarrow{v} \), the vector which has same direction and sense of \( \overrightarrow{x} \) but unitary magnitude. The product \( a \overrightarrow{x} \), of the real number \( a \) and \( \overrightarrow{x} \), is a vector that has same direction of \( \overrightarrow{x} \), same sense of \( \overrightarrow{x} \) if \( a > 0 \) or opposite to that of \( \overrightarrow{x} \) if \( a < 0 \), and magnitude of which \( |a\overrightarrow{x}| = |a||\overrightarrow{x}| \); and from this follows \( \overrightarrow{x} = x\overrightarrow{v}_x \). The scalar product \( \overrightarrow{a} \cdot \overrightarrow{b} \) of the vectors \( \overrightarrow{a} \) and \( \overrightarrow{b} \) is defined by \( \overrightarrow{a} \cdot \overrightarrow{b} = \overrightarrow{a} \overrightarrow{b} \cos \alpha_{\overrightarrow{a}\overrightarrow{b}} \) where \( \alpha_{\overrightarrow{a}\overrightarrow{b}} \) is the smaller angle between \( \overrightarrow{a} \) and \( \overrightarrow{b} \) when these are specified as vectors applied at a same point.

Are placed \( \mathbb{R}^\hat{n} = \prod_{n=1}^{\hat{n}} \mathbb{R}^1 \), \( \mathbb{R}^1 = \mathbb{R} \equiv (-\infty, \infty) \), and thus (referring to section 2.4.1 of [1]) is had \( \mathbb{R}^\hat{n} \rightarrow \mathbb{S}^\hat{n} \) a \( \hat{n} \)-dimensional Euclidean space. It is intended that \( \mathbb{S}^\hat{n} \) is equipped with a orthogonal Cartesian reference system, which has coordinates \( \overrightarrow{x} \) each measured on the respective coordinated axes, and which has coordinated versors \( \overrightarrow{v}_n \), of which \( \overrightarrow{v}_n = \{\overrightarrow{v}_{n}; n = 1, \hat{n}\} \), each having direction and sense of the respective coordinate axis. \( \overrightarrow{x} = \sum_{n=1}^{\hat{n}} x_n\overrightarrow{v}_n \) defines \( \overrightarrow{x} \) as a vector of \( \mathbb{S}^\hat{n} \) which has the components \( x \). Is had \( \overrightarrow{v}_n \cdot \overrightarrow{v}_n = \delta_{nn} \), \( \overrightarrow{x} \cdot \overrightarrow{x} = x \),

\[
\overrightarrow{x} \cdot \overrightarrow{x} = x^2 = \left( \sum_{n=1}^{\hat{n}} x_n\overrightarrow{v}_n \right)^2 = \sum_{n=1}^{\hat{n}} \sum_{n=1}^{\hat{n}} x_n x_n\overrightarrow{v}_n \overrightarrow{v}_n = \sum_{n=1}^{\hat{n}} x_n^2
\]

whose \( x^2 = \sum_{n=1}^{\hat{n}} x_n^2 \) generalizes the Pythagorean theorem.

Are placed \( \overrightarrow{C} = \{\overrightarrow{c}, \overrightarrow{c}\} \subseteq \{\mathbb{S}^1 \rightarrow \mathbb{R}^1\} \) and \( \overrightarrow{C} \subseteq \mathbb{S}^\hat{n} \). These define \( \overrightarrow{C} \) as a curve lying in \( \mathbb{S}^\hat{n} \) and which does not intersect itself. Such \( \overrightarrow{C} \) is identified by its parametric functions \( \overrightarrow{x}(\overrightarrow{c}) \) of which \( \overrightarrow{x}(\overrightarrow{c}) = \{\overrightarrow{x}_n(\overrightarrow{c}); n = 1, \hat{n}\} \) and defined in \( \mathcal{N}(\overrightarrow{c}) \) of which \( \mathcal{N}_c = \{\overrightarrow{c}, \overrightarrow{c}\} \), being \( \overrightarrow{c} \) the curvilinear abscissa measured on \( \overrightarrow{C} \) in the sense that \( \overrightarrow{C} - \overrightarrow{c} \) is the length of the section of \( \overrightarrow{C} \) which has \( \overrightarrow{x}(\overrightarrow{c}) \) and \( \overrightarrow{x}(\overrightarrow{c}) \) as extreme points. These \( \overrightarrow{x}(\overrightarrow{c}) \) constitute the parametric equations of \( \overrightarrow{C} \) i.e. the \( \overrightarrow{x} = \overrightarrow{x}(\overrightarrow{c}) \) of which \( \overrightarrow{x} = \{\overrightarrow{x}_n; n = 1, \hat{n}\} \in \overrightarrow{C} \).

The versor \( \overrightarrow{\tau} \), tangent to \( \overrightarrow{C} \) at the point \( \overrightarrow{c} \), is expressed by \( \overrightarrow{\tau} = \overrightarrow{\tau}(\overrightarrow{c}) = \sum_{n=1}^{\hat{n}} \tau_n(\overrightarrow{c})\overrightarrow{v}_n \) of which \( \tau_n = \overrightarrow{\tau}_n(\overrightarrow{c}) \) with \( \tau_n \) the \( n \)-th direction cosine of the tangent to \( \overrightarrow{C} \) at \( \overrightarrow{c} \) and oriented concordantly with increasing \( \overrightarrow{c} \).

A \( \overrightarrow{C} \) is regular if \( \{\overrightarrow{x}_n(\overrightarrow{c}); n = 1, \hat{n}\} \) and thus, in the graphical aspect, is if, beyond that continuous, also devoid of angular points or cusps where the function \( \overrightarrow{\tau}(\overrightarrow{c}) \) would have a jump.

The said \( \sum_{n=1}^{\hat{n}} x_n^2 = x^2 \) has the specification \( \sum_{n=1}^{\hat{n}} \tau_n^2 = 1 \). Multiplying this by \( (d\overrightarrow{c})^2 \) and considering \( \tau_n = d\overrightarrow{x}_n / d\overrightarrow{c} \) is deduced \( (d\overrightarrow{c})^2 = \sum_{n=1}^{\hat{n}} (d\overrightarrow{x}_n)^2 \) (valid for a regular \( \overrightarrow{C} \)).
Is placed \( f(c) \equiv f(\mathbf{x}(c)) \). From this; the known rules of derivation of a composite function, \( \tau_n = \mathbf{x}'_n \); follows

\[
t'(c) = \frac{df(\mathbf{x}(c))}{dc} = \sum_{n=1}^{\hat{n}} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}_n} \tau_n = \nabla f(\mathbf{x}) \cdot \hat{\tau}
\]

(29)

where \( t'(c) \) is the directional derivative of \( f(\mathbf{x}) \) at the point \( \mathbf{x}(c) \) according to direction and sense of \( \hat{\tau} \), the vector \( \nabla f(\mathbf{x}) \) is the gradient of \( f(\mathbf{x}) \) defined by \( \nabla f(\mathbf{x}) = \sum_{n=1}^{n} \hat{\nu}_n \partial f(\mathbf{x})/\partial x_n \), and of which \( \{\partial f(x)/\partial x_n = \partial f(\mathbf{x})/\partial \mathbf{x}_n; \forall x \equiv \mathbf{x}\} \).

The \( h \)-th derivative of the product of two functions \( \varepsilon(x) \) and \( g(x) \) is expressed by the Leibniz’s rule ([10],[11])

\[
(\varepsilon g)^{(h)} = \sum_{k=0}^{h} \binom{h}{k} (\varepsilon)^{(h-k)} g^{(k)}
\]

(30)

where \( \binom{N}{K} \) is the binomial coefficient defined by

\[
\binom{N}{K} = \frac{N!}{K!(N-K)!} = \frac{\prod_{i=N-K+1}^{N} i}{K!}
\]

The symbol “...” generally implies other symbols considered evident, and in particular when it is inserted in a sequence indicates that this is constituted by elements that vary from the first to the last neatly with the next trend indicated by the first two, thus having \( \{s_1, s_2, \ldots s_k \mid k = 1\} = \{s_1\} \) and \( \{s_1, s_2, \ldots s_k \mid k = 2\} = \{s_1, s_2\} \).

Are placed \( o \geq 1 \) and

\[
f_{(n_1; i=1.a)} = f_{n_1 n_2 \ldots n_a} = \frac{\partial^o f(\mathbf{x})}{\partial \mathbf{x}_{n_1} \partial \mathbf{x}_{n_2} \ldots \partial \mathbf{x}_{n_a}}
\]

from which it follows in particular \( f_n = \partial f(\mathbf{x})/\partial \mathbf{x}_n \).

From: \( o \geq 1; (29); p; \mathcal{A}(\varepsilon, f_n, o - 1 \parallel \varepsilon, g, h \parallel (30)) \); follows

\[
f^{(o)} = (t')^{(o-1)} = (\sum_{n=1}^{\hat{n}} f_n \tau_n)^{(o-1)} = \sum_{n=1}^{\hat{n}} (\tau_n f_n)^{(o-1)} = \sum_{n=1}^{\hat{n}} \sum_{k=0}^{o-1} \binom{o-1}{k} \tau_n^{(o-k-1)} f_n^{(k)}
\]

(31)

which gives rise to

\[
|f^{(o)}| \leq \sum_{n=1}^{\hat{n}} \sum_{k=0}^{o-1} \binom{o-1}{k} |\tau_n^{(o-k-1)} f_n^{(k)}| \leq \Psi\Phi
\]

(32)

of which

\[
\Psi = \sum_{n=1}^{\hat{n}} \sum_{k=0}^{o-1} \binom{o-1}{k} |\tau_n^{(o-k-1)}| \quad \Phi = \max\{|f_n^{(k)}|; k = 0, o - 1; n = 1, \hat{n}\}
\]

17
Is placed $P_r \equiv \{ \tau_n(c) \equiv \tau_n; n = 1, \hat{n} \}$ for which is had $P_r$ if the $\{ \tau_n; n = 1, \hat{n} \}$ are constant as happens when the curve $C$ is a straight line segment. From: $P_r$, first and fourth member of (31); follows IPM

$$ \left\{ |f^{(0)}| = \frac{\hat{n}}{n=1} \sum_{n=1}^{\hat{n}} |\tau_n f_n^{(o-1)}| \leq \sum_{n=1}^{\hat{n}} |\tau_n f_n^{(o-1)}| \leq \Psi_{n_0} |f_{n_0}^{(o-1)}| \right\} \iff P_r \text{ (33)} $$

of which

$$ \Psi_{n_0} \equiv \sum_{n=1}^{\hat{n}} |\tau_n| \quad |f_{n_0}^{(o-1)}| \equiv \max\{|f_n^{(o-1)}|; n = 1, \hat{n}\} $$

The $\Psi_n \leq \Psi_o, \{ f_{n_0}^{(o-1)} \} \leq \hat{\Phi}_o, (32)$ and (33) show how $P_r$ entails an upper bound of $|f^{(0)}|$, i.e. of the maximum absolute value of a derivative defined on a curve, generally less than that implied by $-P_r$.

From: $P_r$, (31); $P_r, \mathcal{A}(\{ f_{n_1}; o - 1 / f^{(0)}, o / (31) \}; \ldots$; follows IPM

$$ \left\{ f^{(o)} = \sum_{n_1=1}^{\hat{n}} \tau_{n_1} f_{n_1}^{(o-1)} = \sum_{n_1=1}^{\hat{n}} \sum_{n_2=1}^{\hat{n}} \tau_{n_1} \tau_{n_2} f_{n_1 n_2}^{(o-2)} = \ldots = \sum_{n_1=1}^{\hat{n}} \sum_{n_2=1}^{\hat{n}} \ldots \sum_{n_\sigma=1}^{\hat{n}} \Theta_{n_1 n_2 \ldots n_\sigma} \right\} \iff P_r \text{ (34) } $$

of which $\Theta_{n_1 n_2 \ldots n_\sigma} \equiv \tau_{n_1} \tau_{n_2} \ldots \tau_{n_\sigma} f_{n_1 n_2 \ldots n_\sigma}$.

The numerosity of the set of all dispositions with repetition of class $K$ of $N$ objects is $N^K$. A $m_{cba}$ is $a$-th element of $b$-th disposition with repetition of class $c$ of $\{ m = \hat{m}, \hat{m} \}$. Is had

$$ \sum_{m_1=\hat{m}}^{\hat{m}} \sum_{m_2=\hat{m}}^{\hat{m}} \ldots \sum_{m_\kappa=\hat{m}}^{\hat{m}} G_{m_1 m_2 \ldots m_\kappa} = \sum_{b=1}^{(\hat{m}-\hat{m}+1)^K} G_{m_{cba}; a=1, \kappa} \text{ (35) } $$

The

$$ \mathcal{A}(\{ n = 1, \hat{n} \}, o, \Theta_{n_1 n_2 \ldots n_\sigma} / \{ m = \hat{m}, \hat{m} \}, K, G_{m_1 m_2 \ldots m_\sigma} / (35) ) $$

to entail

$$ \sum_{n_1=1}^{\hat{n}} \sum_{n_2=1}^{\hat{n}} \ldots \sum_{n_\sigma=1}^{\hat{n}} \Theta_{n_1 n_2 \ldots n_\sigma} = \sum_{b=1}^{\hat{n}_o} \Theta_{n_{oba}; a=1, o} \text{ (36) } $$

where $n_{oba}$ is $a$-th element of $b$-th disposition with repetition of class $o$ of $\{ n = 1, \hat{n} \}$.

From: $P_r$, (34), (36); expression of $\Theta_{n_1 n_2 \ldots n_\sigma}$; follows IPM

$$ \left\{ \begin{align*} |f^{(o)}| & = \sum_{b=1}^{\hat{n}_o} \Theta_{n_{oba}; a=1, o} \leq \sum_{b=1}^{\hat{n}_o} \tau_{n_{oba}} \tau_{n_{oba}} \ldots \tau_{n_{oba}} f_{n_{oba} n_{oba} \ldots n_{oba}} \\ & \leq \Psi_{n_{oba}} \hat{\Phi}_o \text{ (37) } \end{align*} \right\} \iff P_r $$

18
of which
\[ \Psi_{\mathcal{R}0} = \sum_{b=1}^{\hat{n}} \hat\Theta_{\{n\in\mathcal{A};a=1,0\}} \quad \Theta_{\{n\in\mathcal{A};a=1,0\}} = |\tau_{n_1}\tau_{n_2} \ldots \tau_{n_0}| \]
\[ \hat\Phi_{\mathcal{R}0} = \max \left\{ \left| \frac{\partial^2 f(x)}{\partial \mathcal{K}_{n_1} \partial \mathcal{K}_{n_2} \ldots \partial \mathcal{K}_{n_0}} \right| \right\} \quad \{ n_a \in \{ n = 1, \hat{n} \}; a = 1, 0 \} \]

For the $|f^{(o-1)}_{n_0}|$ of (33) is had $\mathcal{E}(\mathcal{O} - 1, f_{n_0} / \mathcal{O}, f / (33))$ which implies
\[ P_{\tau} \Rightarrow \left\{ |f^{(o-1)}_{n_0}| \leq \Psi_0 |f^{(o-2)}_{n_0n_{n-1}}| \right\} \quad |f^{(o-2)}_{n_0n_{n-1}}| = \max \{ f^{(o-2)}_{n_0n_{n-1}}; n = 1, \hat{n} \} \]

This, $P_{\tau}$ and (33) entail $|f^{(o)}| \leq \Psi^2_{\mathcal{R}} |f^{(o-2)}_{n_0n_{n-1}}|$; and, by decreasing subsequently in the same way up to 0 the derivation order of the second member, is reached the first bound of
\[ \left\{ |f^{(o)}| \leq \Psi^2_{\mathcal{R}} |f_{n_0n_{n-1} \ldots n_1}| \leq \Psi^2_{\mathcal{R}} \hat\Phi_{\mathcal{R}0} \right\} \leq P_{\tau} \quad (38) \]
of which $|f_{n_0n_{n-1} \ldots n_1}| \equiv \max \{ f_{n_0n_{n-1} \ldots n_1}; n = 1, \hat{n} \}$.

The comparison between (37) and (38) show $\Psi_{\mathcal{R}0} = \Psi_0^2$ which can be confirmed as follows. From: definition of $\Psi_{\mathcal{R}0}$:
\[ \mathcal{E}(\{ n = 1, \hat{n} \}, \mathcal{O}, \hat\Theta_{\{n\in\mathcal{A};a=1,0\}} / \{ m = \hat{m}, \hat{m} \}, \mathcal{K}, \mathcal{G}_{\{n\in\mathcal{A};a=1,0\}} / (35)); \]
definition of $\hat\Theta_{\{n\in\mathcal{A};a=1,0\}}$; \[ \mathcal{P}; \mathcal{P}$; definition of $\Psi_0$; follows
\[ \Psi_{\mathcal{R}0} = \sum_{b=1}^{\hat{n}} \hat\Theta_{\{n\in\mathcal{A};a=1,0\}} = \sum_{n_1=1}^{\hat{n}} \sum_{n_2=1}^{\hat{n}} \cdots \sum_{n_0=1}^{\hat{n}} \hat\Theta_{n_1n_2 \ldots n_0} = \sum_{n_1=1}^{\hat{n}} \sum_{n_2=1}^{\hat{n}} \cdots \sum_{n_0=1}^{\hat{n}} |\tau_{n_1}\tau_{n_2} \ldots \tau_{n_0}| =
\[ \sum_{n_1=1}^{\hat{n}} |\tau_{n_1}| \sum_{n_2=1}^{\hat{n}} |\tau_{n_2}| \cdots \sum_{n_0=1}^{\hat{n}} |\tau_{n_0}| = \left( \sum_{n=1}^{\hat{n}} |\tau_n| \right) = \Psi^2_{\mathcal{R}} \]

4 The approximation of a linear combination of directional derivatives that expresses a partial derivative at a intersection of some curves

Is considered the set of curves $\{ \mathcal{C}_c; c = 1, \hat{c} \}$ of which $\hat{c} \geq \hat{n},$
\[ \{ \mathcal{E}(\mathcal{C}_{c}, \mathcal{C}_c, \mathcal{K}_c, \mathcal{K}_{cn}, \mathcal{C}_c, \mathcal{C}_c / \mathcal{C}_c, \mathcal{C}_c, \mathcal{K}_c, \mathcal{C}_c, \mathcal{C}_c / (29)); \mathcal{K} = \mathcal{K}_{\mathcal{C}}(\mathcal{C}_c); c = 1, \hat{c} \}
\]

$c_{c1} = \hat{c}_c = 0 \quad c_{ci} = \hat{c}_c \quad \{ c_{ci} < c_{ci+1} ; i = 2, \hat{i}_c \} \quad \hat{c}_c \equiv c_{c\hat{i}_c} \quad \hat{i}_c \in \{ i = 1, \hat{i}_c \}

This gives rise to the linear system $\mathcal{T} \cdot \mathcal{D} = \mathcal{F}$ of which
\[ \mathcal{T} \equiv \left[ \tau_{cn}; c = 1, \hat{c}; n = 1, \hat{n} \right] \quad \tau_{cn} = \mathcal{K}_{cn}(\mathcal{C}_c) \quad \mathcal{D} \equiv \left\{ \frac{\partial f(x)}{\partial \mathcal{K}_n}; n = 1, \hat{n} \right\}
\]
\[ \left\{ \frac{\partial f(x)}{\partial \mathcal{K}_n}; c = 1, \hat{c} \right\} \quad \mathcal{F} \equiv \{ F_c; c = 1, \hat{c} \} \quad F_c = f'_{c}(\mathcal{C}_c) \quad f_c(\mathcal{C}_c) \equiv f(\mathcal{K}_c(\mathcal{C}_c)) \]
Is considered the \( \{ c_{nb}; n = 1, \hat{n}; b = 1, \hat{b} \} \) that verifies

\[
\{ \det T_b \neq 0; b = 1, \hat{b} \} \quad \text{ } T_b \equiv [ \tau_{cn,n}; n = 1, \hat{n}; n = 1, \hat{n} ]
\]

This and (2.3.4) of [1] entail \( \{ D = T_b^{-1} \cdot F_b; b = 1, \hat{b} \} \) of which

\[
T_b^{-1} \equiv [ T_{bnn}; n = 1, \hat{n}; n = 1, \hat{n} ] \quad \text{ } F_b \equiv \{ F_{cab}; n = 1, \hat{n} \}
\]

and which gives rise to

\[
\{ D_{Ab} = T_b^{-1} \cdot F_{Ab}; b = 1, \hat{b} \} \quad \text{ } D_{Ab} \equiv \{ -D_{Abn}; n = 1, \hat{n} \} \quad \text{ } F_{Ab} \equiv \{ -F_{Acab}; n = 1, \hat{n} \}
\]

The \( D = T_b^{-1} \cdot F_b \) and \( D_{Ab} = T_b^{-1} \cdot F_{Ab} \) imply \( D + D_{Ab} = T_b^{-1} \cdot F_b + T_b^{-1} \cdot F_{Ab} \) from which follows

\[
\tilde{E}_b = T_b^{-1} \cdot E_b \quad \tilde{E}_b \equiv D + D_{Ab} \equiv \{ \tilde{E}_{bn}; n = 1, \hat{n} \} \quad \tilde{E}_{bn} = \frac{\partial f(x)}{\partial x_n} - D_{Abn}
\]

\[
E_b \equiv F_b + F_{Ab} \equiv \{ E_{cab}; n = 1, \hat{n} \} \quad \text{ } E_c = F_c - F_{Ac}
\]

Is considered every \( F_{Ac} \) as a known approximation of the unknown \( F_c \), by following, for \( D_{Ab} = T_b^{-1} \cdot F_{Ab} \) and \( D = T_b^{-1} \cdot F_b \), that every \( D_{Abn} \) is a known approximation of the unknown \( \partial f/\partial x_p \), and therefore that \( E_{cab} \) and \( \tilde{E}_{bn} \) are the errors of the respective approximations of \( F_{cab} \) with \( F_{Acab} \) and of \( \partial f/\partial x_p \) with \( D_{Abn} \).

The \( \mathcal{E}(c_{ci}, f_c(c_{ci}); i = 1, \hat{i}_c \parallel x_p, y(x_p); p = 1, \hat{p} \parallel (14)) \) allows to place

\[
F_{Ac} = \sum_{i=1}^{\hat{i}_c} \Lambda_{ci} f_c(c_{ci})
\]

whose \( \{ \Lambda_{ci}; i = 1, \hat{i}_c \} \) are knowable by means of

\[
\mathcal{E}(\tilde{c}_c, \{ c_{ci}; i = 1, \hat{i}_c \} \parallel x_p, \{ x_p; p = 1, \hat{p} \} \parallel (14))
\]

\[
\{ \mathcal{E}(\Lambda_{ci}; i = 1, \hat{i}_c \parallel \lambda_{pp}; p = 1, \hat{p} \parallel (14)); \forall i_c < \hat{i}_c \}
\]

\[
\{ \mathcal{E}(\Lambda_{ci}; i = 1, \hat{i}_c \parallel \delta_{p}; p = 1, \hat{p} \parallel (14)); \forall i_c = \hat{i}_c \}
\]

The \( \mathcal{E}(F_c, F_{Ac}, -\mathcal{E}_c \parallel y'(x_p), \tilde{S}'(x_p), E_c' \parallel \text{ section 2.3}) \), due to (39) and \( \mathcal{E}_c = F_c - F_{Ac} \), entails \( |\mathcal{E}_c| \leq K_c \varphi_c \) whose \( K_c \) is knowable by means of \( \mathcal{E}(-\mathcal{E}_c, K_c, \varphi_c \parallel E_c', K_p, \Phi(y^{(4)}) \parallel (28)) \) and of which \( \varphi_c = \max \{|f_c^{(4)}| \parallel c \in (0, \hat{c}_c)\} \).

This and \( \mathcal{E}(f_c(c_{ci}), 4 \parallel f(c), 0 \parallel (32), (38)) \) allow to place \( \varphi_c = \tilde{\psi}_c \varphi_c \) whose \( \tilde{\psi}_c \) is known. Therefore is had \( |\mathcal{E}_c| \leq \psi_c \varphi_c \) of which \( \psi_c = K_c \psi_c \) and with \( \psi_c \) known.

The \( \tilde{E}_b = T_b^{-1} \cdot E_b \) has the expression \( \{ \tilde{E}_{bn} = T_{bnn} \varphi_c; n = 1, \hat{n} \} \) that, on the basis of \( |\mathcal{E}_c| \leq \psi_c \varphi_c \), gives rise to \( \{ |\tilde{E}_{bn}| \leq \psi_{bn} \varphi_c; n = 1, \hat{n} \} \) of which \( \tilde{\varphi} = \max \{ \varphi_c; c = 1, \hat{c} \} \) and whose \( \tilde{\psi}_{bn} \) is made known by \( \tilde{\psi}_{bn} = \sum_{n=1}^{\hat{n}} |T_{bnn}| \psi_{cab} \).

These bounds imply that the most convenient, among the alternative \( \{ \tilde{E}_{bn}; b = 1, \hat{b} \} \), is the \( \tilde{E}_{bn} \) of which \( B \equiv \{ b \parallel \tilde{\psi}_{bn} = \min \{ \tilde{\psi}_{bn}; b = 1, \hat{b} \} \} \).
From: this and \( \hat{\mathcal{E}}_{bn} = \partial f / \partial x_n - D_{Abn} \); \( D_{Abn} = \sum_{n=1}^{\hat{n}} \mathcal{T}_{bmn} \mathcal{F}_{c_{nbn}} \) (due to \( D_{Ab} = \mathcal{T}_{b}^{-1} \cdot \mathcal{F}_{Ab} \)); (39); \( f_c(C_c) = f(\mathcal{X}_c(C_c)) \); follows

\[
\frac{\partial f(x)}{\partial x_n} = D_{\hat{A}bn} + \hat{\mathcal{E}}_{bn} = \sum_{n=1}^{\hat{n}} \mathcal{T}_{\hat{b}nn} \mathcal{F}_{c_{nbn}} + \hat{\mathcal{E}}_{bn} = \sum_{n=1}^{\hat{n}} \mathcal{T}_{\hat{b}nn} \sum_{i=1}^{i_{c_{nbn}}} \Lambda_{c_{nbi}} f_{c_{nbi}}(C_{c_{nbi}}) + \hat{\mathcal{E}}_{bn} = 
\]

\[
\sum_{n=1}^{\hat{n}} \mathcal{T}_{\hat{b}nn} \sum_{i=1}^{i_{c_{nbn}}} \Lambda_{c_{nbi}} f_{c_{nbi}}(C_{c_{nbi}}) + \hat{\mathcal{E}}_{bn}
\]

that, admitting \( \hat{\mathcal{E}}_{bn} \approx 0 \), gives rise to

\[
\frac{\partial f(x)}{\partial x_n} \approx \sum_{n=1}^{\hat{n}} \mathcal{T}_{\hat{b}nn} \Lambda_{c_{nbi}} f_{c_{nbi}}(C_{c_{nbi}}) \tag{40}
\]

as an approximation of the linear combination of directional derivatives \( \sum_{n=1}^{\hat{n}} \mathcal{T}_{\hat{b}nn} \mathcal{F}_{c_{nbn}} \), which, according to \( \mathcal{D} = \mathcal{T}_{\hat{b}}^{-1} \cdot \mathcal{F}_{\hat{b}} \), expresses \( \partial f(x)/\partial x_n \) at the point \( x \) of which \( \{x = \mathcal{X}_c(\hat{c}_c); c = 1, \hat{c}\} \).

5 The formulation of a differential analytical model and its numerical solution as the unknowns of a total system

The generic differential analytical model \( \mathfrak{M} \), whose solution is (as said in the introduction) the purpose of this paper, has an expression \( \mathfrak{M}(x) \) of which

\[
\mathfrak{M}_x \equiv \{ \mathcal{F}_m(x), \mathcal{D}_m(x), \mathcal{F}_{nm}(x) \} = 0; \ m = 1, \hat{m} \} \quad \mathcal{F}_m(x) \subseteq \mathfrak{M}(x) \quad \mathcal{D}_m(x) \subseteq \mathcal{D}(x) 
\]

\[
\mathcal{F}_{nm}(x) \subseteq \mathcal{F}_n(x) \quad \mathfrak{M}(x) = \{ \mathcal{F}_m(x); m = 1, \hat{m} \} \quad \mathfrak{M}(x) = \{ \mathcal{D}_d(x); d = 1, \hat{d} \} 
\]

\[
\mathcal{F}_d(x) = \{ \mathcal{F}_{nm}(x); m = 1, \hat{m} \} \quad \mathcal{F}_d(x) = \{ \mathcal{F}_d(x); o = 0, \hat{d} \} 
\]

\[
\mathcal{D}_d = \frac{\partial \mathcal{F}_{md}}{\partial x_{n_{d1}} \partial x_{n_{d2}} \ldots \partial x_{n_{d\hat{d}}}} \quad \{ \mathcal{F}_{do} = \frac{\partial \mathcal{F}_{d,o-1}}{\partial x_{n_{do}}} \} = \frac{\partial \mathcal{F}_{md}}{\partial x_{n_{d1}} \partial x_{n_{d2}} \ldots \partial x_{n_{d\hat{d}}}}; o = 1, \hat{d} \}
\]

where: \( \mathcal{F}(x) \) are unknown functions which appear autonomously (i.e. as derivatives of order 0) or in \( \mathcal{D}(x) \) as functions to be derived; \( \mathcal{D}(x) \) are unknown functions, since they are derivatives whose functions to be derived are the unknown \( \mathcal{F}(x) \); \( \mathcal{F}_n(x) \) are functions whose values are known at the points of \( \mathfrak{R}(x) \) where is desired to know the \( \mathcal{F}(x) \); each \( \{ n_{do}; o = 1, \hat{o} \} \) (of which \( 1 \leq o \leq \hat{o} \)) is a combination generally with repetition of class \( o \) of the \( \{ n = 1, \hat{n} \} \) and is replaceable by an its permutation as enables the Schwarz’s theorem ([10],[11]); \( x \) is arbitrary unless \( x \in \mathfrak{R}_x \).

The knowledge of \( \mathcal{F}(x) \) can be considered prevented by not knowing \( \mathcal{D}(x) \), in the sense that, if these were functions known of \( \mathcal{F}(x) \), \( \mathfrak{M}(x) \) would be solved as a system
of \( \hat{m} \) non differential equations in the \( \hat{m} \) unknowns constituted by the \( F(\underline{x}) \). Therefore a solution of \( \mathfrak{M} \), numerical as it is understood in the following, can be obtained by removing this obstacle by means of approximating the values of \( D(\underline{x}) \) with known functions of values of \( F(\underline{x}) \).

One such solution is a set of numbers called \( \underline{s} \), of which

\[
\underline{s} \equiv \{ f_{mp}; m = 1, \hat{m}; p = 1, \hat{p} \} \quad f_{mp} \approx F_m(\underline{x}_p) \quad \underline{x}_p \in \mathfrak{R} \;
\]

\[
\mathfrak{K} \equiv \{ \underline{x}_p; p = 1, \hat{p} \} \subset \mathfrak{R} \quad 1 \leq \hat{p} \neq \infty
\]

In seeking a \( \underline{s} \), are given as known \( \mathfrak{R} \), \( \mathfrak{K} \), \( \{ F_m(\underline{x}_p); p = 1, \hat{p} \} \) and contingently a set of conditions \( \mathfrak{C} \) of which

\[
\mathfrak{C} \subset \{ \{ F_m(\underline{x}_p) - \tilde{F}_{mp} = 0; m = 1, \hat{m} \},
\{ \tilde{F}_{do}(\underline{x}_p) - \tilde{F}_{dop} = 0; o = 1, \tilde{o}_d; d = 1, \hat{d} \}; p = 1, \hat{p} \} \quad (41)
\]

where each \( F_m(\underline{x}_p) - \tilde{F}_{mp} = 0 \) means that \( F_m(\underline{x}) \) has at \( \underline{x}_p \) the known value \( \tilde{F}_{mp} \), and similarly for each \( \tilde{F}_{do}(\underline{x}_p) - \tilde{F}_{dop} = 0 \).

These \( \mathfrak{C} \) imply the eventuality that at some \( \underline{x}_p \) become all known the \( F(\underline{x}_p) \), i.e. the chance of some relations of type

\[
\{ F_m(\underline{x}_p) - \tilde{F}_{mp} = 0; m = 1, \hat{m} \} \subset \mathfrak{C}
\]

and therefore rise to a

\[
\mathfrak{K} \equiv \{ \underline{x}_p; p = 1, \hat{p} \} \equiv \{ \underline{x}_{p0}; p = 1, \hat{p} \} \subset \mathfrak{K}
\]

such that the numerosity of \( \mathfrak{K} \) is the maximum compatibly with being at each \( \underline{x}_p \) unknown at least one of the \( F(\underline{x}_p) \). Moreover is had also the eventuality that the \( \mathfrak{C} \) do take the form \( 0 = 0 \) to some equations of a \( \mathfrak{M}(\underline{x}_p) \).

Is called \( \mathfrak{R} \), the system of \( \hat{m}_p \) equations constituted by the

\[
\mathfrak{C} \cap \{ \tilde{F}_{do}(\underline{x}_{p0}); p = 1, \hat{p} \} = 0; o = 1, \tilde{o}_d; d = 1, \hat{d} \}
\]

and by those that are obtained by removing from \( \mathfrak{M}(\underline{x}_p) \) the equations that the \( \mathfrak{C} \) reduce to the useless form \( 0 = 0 \).

A such \( \mathfrak{R} \) can be lacking in some of the \( \{ \{ F_m(\underline{x}_p); m = 1, \hat{m} \}, \{ D_{do}(\underline{x}_{p0}); d = 1, \hat{d} \} \} \), may include some of the \( \mathfrak{C} \) and is had

\[
\mathfrak{R} \equiv \{ F_{pm} \{ F_{pm}; D_{pm}(\underline{x}_p); m = 1, \hat{m}_p \} \} \quad F_{pm} \subset F_p \equiv \{ F_{pm}; m = 1, \hat{m}_p \} \subset \mathfrak{F}(\underline{x}_p)
\]

\[
D_{pm}(\underline{x}_p) \subset D_{pd}(\underline{x}_p) \equiv \{ D_{pd}(\underline{x}_p); d = 1, \hat{d}_p \} \quad \hat{m}_p \leq \hat{m}_p \quad \hat{o}_{pd} > 0 \quad \hat{m}_{pd} \in \{ m = 1, \hat{m} \}
\]

\[
\mathfrak{F}_{pd} \equiv \{ \mathfrak{F}_{pdo}; o = 0, \hat{o}_{pd} \} \quad \mathfrak{F}_{pdo} \equiv F_{mpd} \quad \mathfrak{F}_{pdo} \equiv D_{pd}
\]

\[
D_{pd} \equiv \frac{\partial^{\hat{m}_{pdo}}F_{mpd}}{\partial x_{n_{pd1}} \partial x_{n_{pd2}} \ldots \partial x_{n_{pd0}}}
\]

\[
\{ \mathfrak{F}_{pdo} \equiv \frac{\partial^{\hat{o}_{pdo}}F_{mpd}}{\partial x_{n_{pdo1}} \partial x_{n_{pdo2}} \ldots \partial x_{n_{pdo}}}; o = 1, \hat{o}_{pd} \} \quad (42)
\]
where: $F_p$ are unknown and appear autonomously or as functions to be derived; $D_p(\tilde{x}_p)$ are all treated as unknown even if some of them are made known by $\mathcal{C}$; each $\{n_{p\delta o}; o = 1, \delta_{pd}\}$ is a combination generally with repetition of class $\tilde{o}_{pd}$ of the $\{n = 1, \tilde{n}\}$.

Therefore the total system $\mathcal{MM}$, of which $\mathcal{MM} \equiv \{\mathcal{M}_p; p = 1, \tilde{p}\}$, is a system of $\mathcal{N}_e$ equations, of which $\mathcal{M}_e = \sum_{p=1}^{\tilde{p}} \tilde{m}_p$, where are unknown $\{D_p(\tilde{x}_p); p = 1, \tilde{p}\}$ and the $\mathcal{M}_i$ values $\{F_p; p = 1, \tilde{p}\}$ of which $\mathcal{N}_i = \sum_{p=1}^{\tilde{p}} \tilde{m}_p \leq \mathcal{N}_e$. Expressing then each of $\{D_{pd}(\tilde{x}_p); d = 1, \delta_p; p = 1, \tilde{p}\}$ by means of a respective

$$D_{pd}(\tilde{x}_p) = \tilde{D}_{pd}(\tilde{f}_{pd}, \tilde{E}_{Tpd}) \equiv D(\tilde{f}_{pd}) + \mathcal{E}_{Tpd} \approx D(\tilde{f}_{pd})$$

of which $\tilde{f}_{pd} \equiv \{f_{pd i}; i = 1, \tilde{p}_d\} \subseteq \{f_{pm}(\tilde{x}_p); p = 1, \tilde{p}\}$, $\tilde{f}_{pd} \subseteq \{F_p; p = 1, \tilde{p}\}$, and where $D(\tilde{f}_{pd})$ is a known function that approximates $\tilde{D}_{pd}(\tilde{f}_{pd}, \tilde{E}_{Tpd})$ with an error $\mathcal{E}_{Tpd}$ of which is admitted $\mathcal{E}_{Tpd} \approx 0$; $\mathcal{MM}$ becomes a system $\{E_{Ts}(F_p; p = 1, \tilde{p}) = 0; s = 1, \mathcal{N}_e\}$, nondifferential and generally nonlinear, of $\mathcal{N}_e$ equations in the $\mathcal{N}_i$ unknowns $\{F_{pm}; m = 1, \tilde{m}_p; p = 1, \tilde{p}\}$, and that according to $\mathcal{N}_i \leq \mathcal{N}_e$ can be solved with the known methods of numerical analysis such as that of Newton-Raphson reported in section 2.4.5 of [1].

Indeed this method is feasible by calculating some successive solutions of a linear system and, in the case, as is $\mathcal{MM}$, of a number of equations $\mathcal{N}_e$ not less than the number of unknowns $\mathcal{N}_i$, each of these solutions is obtainable by applying the (2.3.4) of [1], and in particular using the Gauss’s method with the variant of “maximum pivot” and with the only modification of considering all the $\mathcal{N}_e$ equations even if the implicated pivots are only $\mathcal{N}_i$.

In seeking a $\hat{\mathcal{S}}$, as the $\mathcal{C}$ are contingently known other conditions, each consisting of an equation similar to those of $\mathcal{MM}$, i.e. of the type $A(\mathcal{B}(\mathcal{X}), \mathcal{C}(\mathcal{X}), \mathcal{X}) = 0$ of which $\mathcal{B}(\mathcal{X}) \subseteq \mathcal{F}(\mathcal{X})$ and $\mathcal{C}(\mathcal{X}) \subseteq \mathcal{D}(\mathcal{X})$, but imposed only on some of the $\tilde{\mathcal{X}}$ i.e. only on the elements of a $\hat{\mathcal{X}}$ of which $\tilde{\mathcal{X}} \supset \hat{\mathcal{X}}$. Each of these other conditions can be introduced as follows: are added to $\mathcal{F}(\mathcal{X})$ an auxiliary unknown function $F_\Lambda(\mathcal{X})$ and to $\mathcal{M}$ the corresponding equation multiplied by $F_\Lambda(\mathcal{X})$; are added to $\mathcal{C}$ the conditions constituted on having $F_\Lambda(\mathcal{X})$ the values respectively 1 and 0 in the elements of $\tilde{\mathcal{X}}$ and $\hat{\mathcal{X}} - \tilde{\mathcal{X}}$.

6 The approximation of a derivative of the total system with a linear combination of local values of the function to be derived

6.1 The set of rectilinear segments

In relation to $\tilde{\mathcal{X}} \equiv \{x_p; p = 1, \tilde{p}\}$ said in section 5, are placed

$$x_p \equiv \{x_{pn}; n = 1, \tilde{n}\}, \quad \{a, b\} \subseteq \{p = 1, \tilde{p}\}, \quad d_{ab} = \sqrt{\sum_{n=1}^{\tilde{n}} (x_{bn} - x_{an})^2}$$

$$\tau_{ab} \equiv \sum_{n=1}^{\tilde{n}} \frac{x_{bn} - x_{an}}{d_{ab}}$$

23
so $d_{ab}$, $\tilde{\tau}_{ab}$, $\tau_{ab}$ are respectively the distance between $\vec{x}_a$ and $\vec{x}_b$, the versor of the straight line through these and oriented from $\vec{x}_a$ to $\vec{x}_b$, the $n$-th cosine director of such line.

It is understood that from $\mathbb{X}$ is deducible a set $\mathbb{R}$ of straight line segments, whose numerosity is maximum compatibly with the

$$\mathbb{R} \equiv \{ R_r; r = 1, \hat{R} \} \quad \hat{R} \geq 1 \quad \{ \exists x_p \in R_r; p = 1, \hat{P} \} \quad \mathcal{E}(R_r, r_r, \hat{r}_r, R_r / \mathbb{C}, \mathbb{C}, \mathbb{C} / \text{sec. 3})$$

$$r_r = R_{r+1} = 0 \quad \exists r_r = d_{ab} \quad \mathcal{P}_r \equiv \{ x_{p,r}; i = 1, \hat{i}_r \} = R_r \cap \mathbb{X} \quad \hat{i}_r \geq 3$$

$$\{ R_{r,i} < R_{r,i+1}; i = 1, \hat{i}_r - 1 \} \quad R_{r,i} = d_{p_{r,i}r_{r,i}}$$

(44)

Are considered the $\{ x_k; k = 1, \hat{k} \}$ of which $x_1 \neq x_{\hat{k}}$. The condition for which these $\hat{k}$ points lie on a same straight line is expressed by $\{ \tau_{1k} = \omega_k \tau_{1k}; k = 2, \hat{k} - 1 \}$, of which $\omega_k \equiv \pm 1$ and

$$\{ \tau_{1k} = \omega_k \tau_{1k}; k = 2, \hat{k} - 1 \}$$

$$\{ \{ \tau_{1k} = \omega_k \tau_{1k}; n = 1, \hat{n} \} \lor \{ \tau_{1k} = -\tau_{1k}; n = 1, \hat{n} \}; k = 2, \hat{k} - 1 \}$$

(45)

on the basis of

$$\{ \tau_{1k} = \omega_k \tau_{1k} \} \equiv \{ \tau_{1k} \cdot \vec{u}_n = \omega_k \tau_{1k} \cdot \vec{u}_n; n = 1, \hat{n} \} \lor \{ \tau_{1k} = \omega_k \tau_{1k} ; n = 1, \hat{n} \} \equiv \{ \tau_{1k} = \tau_{1k}; n = 1, \hat{n} \} \lor \{ \tau_{1k} = -\tau_{1k}; n = 1, \hat{n} \}$$

The knowledge of $\mathcal{P}_r; r = 1, \hat{R}$ can be achieved with the following algorithm. Are placed $\hat{R} = 0$,

$$\mathcal{A}_{ab\hat{R}} \equiv \{ \{ \tau_{ak} = \omega_k \tau_{ab}; k = \alpha_r, \beta_r \}; r = 1, \hat{R} \} \quad \mathcal{B}_{ab} \equiv \exists \{ \tau_{ak} = \omega_k \tau_{ab} \mid x_k \in \mathbb{X} - \{ x_a, x_b \} \}$$

and are carried out the $\binom{\hat{R}}{2}$ iterations $\{ I_{ab}; b = a + 1, \hat{P}; a = 1, \hat{P} - 1 \}$. For each $I_{ab}$, if

$$\{ \{ \mathcal{A}_{ab\hat{R}} \mid \hat{R} > 0 \} \land \mathcal{B}_{ab} \} \lor \{ \{ \hat{R} = 0 \} \land \mathcal{B}_{ab} \}$$

(this condition can be controlled by means of (45)) $\hat{R}$ is incremented of 1 and are placed $\alpha_{\hat{R}} = a$, $\beta_{\hat{R}} = b$. After these iterations are known the $\{ \alpha_r, \beta_r; r = 1, \hat{R} \}$ and are carried out the iterations indicated by $\{ r = 1, \hat{R} \}$. For each $r$, is placed $\hat{i}_r = 0$ and are carried out the iterations indicated by $\{ p = 1, \hat{P} \}$. For each $p$, if

$$\{ \tau_{\alpha, p} = \omega_p \tau_{\alpha, \beta, r} \} \land \{ p \notin \{ \alpha_r, \beta_r \} \}$$

then $\hat{i}_r$ is incremented of 1 and is placed $P_{r, \hat{i}r} = p$. After these $\hat{P}$ iterations, is modified the subsequent order of the elements of $\mathcal{P}_{r, i}$ so as to have

$$\exists d_{r, i} \equiv \max \{ d_{r,a}p_{r,b}; a = 1, \hat{i}_r; b = 1, \hat{i}_r \} \quad \{ d_{r, i} < d_{r, i+1}; i = 2, \hat{i}_r - 1 \}$$

At the end of this algorithm is known the $\{ P_{r, i}; i = 1, \hat{i}_r; r = 1, \hat{R} \}$ that makes known $\{ \mathcal{P}_r; r = 1, \hat{R} \}$. 

24
6.2 An original algorithm that expresses, by means of a tree graph, the linear combination that approximates a derivative of the total system.

An approximation (43) of a $D_{pd}(\overline{x}_p)$ (of which (42)) can be obtained using as a logical schema an oriented tree graph implemented by means of

$$\{A \{\bar{p}_{pd,0-1}(x) \parallel f(x) \parallel (40)); o = 1, \bar{o}_{pd}\}$$

and of the segments $\mathfrak{R}$ of section 6.1.

A graph $\mathcal{G}$, of which $\mathcal{G} \equiv \{\check{n}, \bar{A}\}$, is constituted by a set of nodes $\check{n}$ of which $\check{n} \equiv \{\check{n}_0; \check{n}_1 = 1, \check{n}_q\}, \check{n}_0 = \infty$, and by a set of arcs $\bar{A}$ of which $\bar{A} \equiv \{\check{a}_a; \check{a} = 1, \bar{a}\}, \check{a} \neq \infty, \check{a}_a \equiv (\check{n}_a, \check{n}_b), \check{n}_a \in \check{n}, \check{n}_b \in \check{n}$.

An arc $(\check{n}_a, \check{n}_b)$ is directional since it identifies the nodes $\check{n}_a$ and $\check{n}_b$, respectively as the origin and destination of an inherent way.

A $\mathcal{G}$ is connected if $\{\exists \mathcal{C}; \forall \{\check{n}_{i,1}, \check{n}_{i,2}\} \in \check{n}\}$. A $\mathcal{G}$ is oriented or not oriented, respectively if $(\check{n}_a, \check{n}_b) \neq (\check{n}_b, \check{n}_a)$ or $(\check{n}_a, \check{n}_b) \equiv (\check{n}_a, \check{n}_b)$. A $\mathcal{G}$ is a tree if $\check{a} = \check{n} - 1$ and is connected the graph that is deduced from it by adding to each $(\check{n}_a, \check{n}_b)$ a respective $(\check{n}_b, \check{n}_a)$. A oriented tree $\mathcal{G}$ has a root node $\check{n}_a$, of which $\check{n}_a \in \check{n}$, that verifies $\{\check{n} \in \check{n} - \check{n}_r\} \Rightarrow \exists \{\check{C} || \{\check{n}_{i,1}, \check{n}_{i,2}\} \equiv \{\check{n}_a, \check{n}_b\}\}$ or

$$\{\check{n} \in \check{n} - \check{n}_r\} \Rightarrow \exists \{\check{C} || \{\check{n}_{i,1}, \check{n}_{i,2}\} \equiv \{\check{n}_a, \check{n}_n\}\} \quad (46)$$

A $\check{n}_a \equiv \check{s}$ affirms that the object $\check{s}$ is associated with the node $\check{n}_a$. The oriented tree $\mathcal{G}$, which is used to obtain an approximation (43) of a $D_{pd}(\overline{x}_p)$, is in particular specified, besides that by $\check{a} = \check{n} - 1$ and (46), by

$$\check{n} = \{\check{n}_q; q = 0, \check{o}_{pd}\} \quad \check{n}_q \equiv \{\check{n}_{q,n}; \check{n} = 1, \check{n}_q\} \quad \check{n}_0 = \{\check{n}_{0,0} \equiv \check{n}_r\} \quad \check{n}_q = \{\check{n}_{q,0} \equiv \check{n}_{o,0}\}$$

$$\{\neg \exists \{\check{A} || \{\check{n}_a, \check{n}_b\} \in \check{n}_q\}; q = 0, \check{o}_{pd}\} \quad \{\neg \exists \{\check{A} || \{\check{n}_a, \check{n}_b\} \in \check{n}_q\}; q = 0, \check{o}_{pd} - 1\}$$

$$\check{A} = \{(\check{n}_{q,n}, \check{n}_{q,n_0}); \check{n}_q = 1, \check{n}_q; n = 1, \check{n}; \check{n} = 1, \check{n}_q; q = 0, \check{o}_{pd} - 1\}$$

and (with reference to (42) and $\overline{x}_p \equiv x_{p,p}$) by $\check{n}_{0,0} = D_{pd}(\overline{x}_{p_0}), \check{p}_{0,0} = p_p$,

$$\{\check{n}_{q,n} = \check{F}_{pd,0_{pd} - q}(x_{p,n}); \check{n} = 1, \check{n}_q; q = 0, \check{o}_{pd} - 1\}$$

$$\{\check{n}_{q,o_{pd} - n} = \check{F}_{m_{pd}}(x_{p,o_{pd} - n}); \check{n} = 1, \check{n}_q; q = 0, \check{o}_{pd} - 1\}$$

of which $\{\check{p}_{q,n} \in \{p = 1, \check{p}\}; \check{n} = 1, \check{n}_q; q = 0, \check{o}_{pd}\}$ and $\check{x} \equiv \{x_p; p = 1, \check{p}\}$, and (with reference to (41) and (44)) by

$$\{\{\check{F}_{pd,0_{pd} - q}(x_{p,n}) - \check{F}_{pd,0_{pd} - q}(\check{x}_{p,n}) = 0\} \equiv \neg \exists (\check{n}_{q,n}, \check{n}_b) \in \check{A}; \check{n} = 1, \check{n}_q; q = 1, \check{o}_{pd} - 1\}$$

$$\{\{\check{n}_{q,n_0} = \check{F}_{pd,0_{pd} - q_1}(\check{x}_{p,n_0}); \check{n}_q = 1, \check{n}_q; \check{n}_q = 1, \check{n}_q; q = 0, \check{o}_{pd} - 1\}$$
of which $r_{\tilde{q}q'n}$ $\in \{r = 1, \tilde{r}\}$, $\mathfrak{M} \equiv \{R_r; r = 1, \tilde{r}\}$, $\cap_{n=1}^\tilde{n} R_{r_qn} = x_{p_{q'n}}$.

The identification of the $\mathcal{C}$ in question is completed by

$$\{ \mathcal{A}_r(x_{p_{q'a}}) \times n_{pd}, q_{pd} = q-1, x_{p_{q'n}} ; \eta = 1, \tilde{r}_{q'n} ; R_{r_qn} ; n = 1, \tilde{n} \} \parallel$$

$$\partial f(x_{n}) \times n,\left\{ \{ t_{c_{n_{b}}}(C_{c_{n_{b}}}) ; i = 1, \tilde{i}_{c_{n_{b}}}, C_{c_{n_{b}}} ; n = 1, \tilde{n} \} \parallel (40) ; \hat{n} = 1, \tilde{n}_{q'q} = 0, o_{pd} - 1 \right\}$$

that gives rise to

$$\left\{ \hat{f}_{pd}, o_{pd} - q(x_{p_{q'a}}) \right\} \times n_{pd}, q_{pd} = q-1, x_{p_{q'n}} \right\}$$

where each $\mathcal{A}_{q'n_{q'}}$ is known as a specification of the $t_{b_{nn}}A_{c_{b_{a}}}i$ of (40), having

$$\left\{ \{ \hat{t}_{q'n_{q'}} = \mathcal{A}_{q'n_{q'}} , \eta = 1, \tilde{t}_{q'n_{q'}} , \hat{t}_{r_{q'n} ; n = 1, \tilde{n} ; \hat{n} = 1, \tilde{n}_{q'q} = 0, o_{pd} - 1 \right\}$$

In what has just been said, is used the set of straight line segments $\mathfrak{M}$ instead of a set of curves generally not straight, because (as said in section 3) the maximum absolute value of a derivative defined on a curve is generally less if this is a straight segment, and therefore is generally less also the inherent error (of the type (28)) that influences (as said in section 4) the previous approximation.

Inherently the oriented tree graph $\mathcal{C}$, is considered the set $\{ \mathcal{C}_{k} ; k = 1, \tilde{k} \}$ of every path $\mathcal{C}_{k}$ of which $\sim 3 \mathcal{C} \supset \mathcal{C}_{k}$. Each $\mathcal{C}_{k}$ is a path that goes from $\tilde{n}_{n}$ to a node of $\tilde{n}_{p_{pd}}$ to which is associated an element of $f_{pd}$ (argument of (43)) or that goes from $\tilde{n}_{n}$ to a node of $\{ \tilde{n}_{q} ; q = 1, o_{pd} \}$ to which is associated the known value of an element of $\mathcal{C}$ (of which (41)). Therefore is had

$$\{ \mathcal{C}_{k} ; k = 1, \tilde{k} \} = \{ \{ \mathcal{C}_{k_{a}} ; a = 1, \tilde{a} \}, \{ \mathcal{C}_{q_{b}} ; b = \tilde{a} + 1, \tilde{k} \} \}$$

whose $\mathcal{C}_{k_{a}}$ and $\mathcal{C}_{q_{b}}$ are respectively of the two types just now said, and thus the searched approximation (43) of $D_{pd}(x_{p_{a}})$ can be the

$$D_{pd}(x_{p_{a}}) \approx \mathcal{C}_{pd} + \sum_{a=1}^{\tilde{a}} \tilde{A}_{k_{a}} \mathcal{F}_{m_{pd}}(x_{p_{ada}})$$

of which

$$\mathcal{C}_{pd} = \sum_{b=\tilde{a}+1}^{\tilde{k}} \tilde{A}_{k_{b}} \mathcal{C}_{k_{b}} \quad x_{p_{ada}} \in \mathfrak{X} \quad \mathcal{F}_{m_{pd}}(x_{p_{ada}}) \in \mathfrak{F}_{pd}$$

where $\mathfrak{C}_{k_{b}}$ is one of the values known in $\mathfrak{C}$ and $\tilde{A}_{k}$ (of which $k = k_{a} \vee k_{b}$) is the product of all factors of type $\mathcal{A}_{q'r_{n_{q'}}}$ that are associated with the nodes $\mathcal{C}_{k}$.

Substituting $\mathcal{F}_{m_{pd}}(x_{p_{ada}})$ with $\sum_{p=1}^{\rho} \delta_{p_{p_{p_{ada}}} \mathcal{F}_{m_{pd}}(x_{p_{p_{ada}}})}$ in the previous approximation of $D_{pd}(x_{p_{a}})$, is had

$$D_{pd}(x_{p_{a}}) \approx \mathcal{C}_{pd} + \sum_{p=1}^{\rho} \tilde{A}_{p_{pd}} \mathcal{F}_{m_{pd}}(x_{p_{a}})$$

(47)
of which $\tilde{\Lambda}_{pd} \equiv \sum_{a=1}^{\tilde{n}} \tilde{\Lambda}_{ka} \delta_{pda}$ and whose $\{\tilde{\Lambda}_{pd}; p = 1, \hat{p}\}$ and $\tilde{\mathbf{c}}_{pd}$ can be made known by an algorithm based on

$$\mathcal{A}(\mathbf{x}_p; \mathbf{R}_{\hat{p}nn}; n = 1, \hat{n}) \parallel \mathbf{x}, \{\tilde{\mathbf{c}}_{na}; n = 1, \hat{n}\} \parallel (40)$$

which gives rise to

$$\frac{\partial f(\mathbf{x}_p)}{\partial x_n} \approx \sum_{n=1}^{\hat{n}} \sum_{i=1}^{i_{pnn}} \lambda_{pnni} f_{\hat{p}nn}(r_{\hat{p}nni})$$

where each $\lambda_{pnni}$ is known and (with reference to (44)) relating to the point $\mathbf{x}_{\hat{p}nni}$. Such algorithm is written as follows in a pseudo-language derived from the Visual Basic:

- $\{B_p = 0; p = 1, \hat{p}\}$
- $N = n_{pd}\tilde{o}_{pd}$

**For** $n = 1$ **To** $\hat{n}$

- $\tilde{r} = \tilde{r}_{p,nn}$
  - **For** $i = 1$ **To** $\hat{i}_r$
    - $B_{pr_i} = \lambda_{pnni}$
  - **Next** $i$
- **Next** $n$

$\tilde{\mathbf{c}}_{pd} = 0$  $\text{IB} \equiv \text{F}$
- **For** $q = 1$ **To** $\tilde{o}_{pd} - 1$
  - $N = n_{pd}o_{pd} - q$
  - **If** $\text{IB} \equiv \text{F}$ **Then**
    - $\text{IB} \equiv \text{T}$
    - **Call** SubrA($q, N, \{A_p; p = 1, \hat{p}\}, \{B_p; p = 1, \hat{p}\}, \tilde{\mathbf{c}}_{pd}$)
  - **Else**
    - $\text{IB} \equiv \text{T}$
    - **Call** SubrA($q, N, \{B_p; p = 1, \hat{p}\}, \{A_p; p = 1, \hat{p}\}, \tilde{\mathbf{c}}_{pd}$)
  - **Next** $q$
- **If** $\text{IB} \equiv \text{T}$ **Then**
  - **Call** SubrB($\{B_p; p = 1, \hat{p}\}, \tilde{\mathbf{c}}_{pd}, \{\tilde{A}_{pd}; p = 1, \hat{p}\}$)
- **Else**
  - **Call** SubrB($\{A_p; p = 1, \hat{p}\}, \tilde{\mathbf{c}}_{pd}, \{\tilde{A}_{pd}; p = 1, \hat{p}\}$)
- **End If**
- **Sub** SubrA($q, N, \{A_p; p = 1, \hat{p}\}, \{B_p; p = 1, \hat{p}\}, \tilde{\mathbf{c}}_{pd}$)
- $\{A_p = 0; p = 1, \hat{p}\}$

27
For $p = 1$ To $\hat{p}$
   If $\{\mathcal{F}_{p}^{d} - \mathcal{F}_{p}^{d} = 0\} \in \mathcal{C}$ Then
   $\mathcal{C}_{pd} = \mathcal{C}_{pd} + B_{p} \mathcal{F}_{p}^{d, \tilde{o}_{pd}-q.p}$
   Else
   For $n = 1$ To $\hat{n}$
     $\tilde{r} = \tilde{r}_{pnn}$
     For $i = 1$ To $\hat{i}_{f}$
       $\mathcal{A}_{p,i} = \mathcal{A}_{p,i} + B_{p} \lambda_{pni}$
       Next $i$
     Next $n$
   End If
   Next $p$
End Sub
Sub SubrB($\{\mathcal{A}_{p}; p = 1, \hat{p}\}, \mathcal{C}_{pd}, \{\hat{\mathcal{A}}_{pdp}; p = 1, \hat{p}\}$)
For $p = 1$ To $\hat{p}$
   If $\{\mathcal{F}_{mp}^{d}(\mathcal{x}_{p}) - \mathcal{F}_{mp}^{d} = 0\} \in \mathcal{C}$ Then
   $\mathcal{C}_{pd} = \mathcal{C}_{pd} + \mathcal{A}_{p} \mathcal{F}_{mp}^{d}$
   $\hat{\mathcal{A}}_{pdp} = 0$
   Else
   $\hat{\mathcal{A}}_{pdp} = \mathcal{A}_{p}$
   End If
Next $p$
End Sub

7 Conclusion

The definitions, procedures and results presented in this paper have been used to implement a computer program to which has been given the name PEEI (as acronym for “Programma agli Elementi di Estensione Infinitesima”) and which is aimed at numerically solve any differential analytical model.

More precisely, this program calculates (if is not impossible) a numerical solution of every system of partial differential equations, with number of equations not lower than that of its functions unknowns, and subjected to any additional condition like those initial or boundary.

The program PEEI is freeware and available in [http://www.giacomo.lorenzoni.name/peei/](http://www.giacomo.lorenzoni.name/peei/) together with numerous application examples that confirm its reliability through the comparison of its solutions with those exact.
References


