A method to numerically solve every differential analytical model

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Abstract

A differential analytical model is a system of so many PDEs in the same (or lesser) number of unknown functions where each PDE is of any order and can be nonlinear. Usually the operative interest of such a model is to determine an its solution when is subject to additional conditions like those boundary or initial.

This work exposes the mathematical basis of a program (freeware in *http://www.giacomo.lorenzoni.name/peei/*) to numerically solve every differential analytical model with every set of additional conditions. In particular is exposed what follows.

Are described the analytical properties of two well known models to approximate a function: the interpolating polynomial and the cubic spline. The values of a natural cubic spline and of its derivatives, in the interpolation nodes, are expressed as linear combinations of the known values of the function to be interpolated and whose coefficients depend only on the nodes. Are obtained new bounds for the errors of a cubic spline. Are presented essential aspects of a curve in a multidimensional Euclidean space, in order to obtain an upper bound for the absolute maximum value of a derivative defined on a curve. Is shown the expression of a partial derivative as a linear combination of directional derivatives and is deduced its optimal approximation. Is formulated the expression of the generic differential analytical model, is identified the main impediment to knowledge of an its exact solution in not knowing its partial derivatives, is circumstantiated the context of information contingently available and is showed how, solving an inherent system of nonlinear equations, can be calculated an its numerical solution. Is exposed an original algorithm that, in this system of nonlinear equations, expresses a derivative as a linear combination of unknowns.

Key words: numerical solution of PDEs systems, differential analytical models, numerical differentiation, splines, graph algorithms, interpolation
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Contents

1 Introduction

This paper concerns the generic differential analytical model $\mathfrak{M}$, intended as a system of so many equations in the same or lesser number of unknowns where each equation is a PDE (partial differential equation) of any order and generally nonlinear, and where the unknowns are functions of same independent variables.

The unknown functions of $\mathfrak{M}$ are its exact solution. A discrete solution of $\mathfrak{M}$ is constituted by a limited number of values that its unknown functions assume in their domain of definition. A numerical solution is an approximation of a discrete solution.

In relation to $\mathfrak{M}$, usually is had the operating purpose of determine an its numerical solution when it is specifically subjected to additional conditions such as those initial or boundary. So in particular the objective of what follows is to determine a numerical solution of $\mathfrak{M}$ when is subject to any set of additional conditions.

The symbology of a technical writing aims to make the comprehension univocal and shorten the exposure. Therefore, with reference to section 2 of [1] (this work is also available at *http://www.giacomo.lorenzoni.name/arganprobstat/*), are premised the following definitions.

A same object has some names each of which gives to it some properties. An $\text{A} \equiv \text{B}$ states that A and B are two names of a same object. In identifying the members of an expression, each " $\equiv$ " is considered at last coherently with the parentheses (and analogously " $\not\equiv$ ", " $=$ ", " $\neq$ "). Is intended $\text{A}\langle\text{B}\rangle \equiv \text{A}_\text{B}$, $\bigwedge \equiv$ AND $\equiv$ "conjunction", $\bigvee \equiv$ OR $\equiv$ "inclusive disjunction", $\veebar \equiv$ XOR $\equiv$ "exclusive disjunction". Being $\mathcal{P}$, $\mathcal{P}_\text{A}$ and $\mathcal{P}_\text{B}$ three propositions, $\neg\mathcal{P}$ is the proposition true if $\mathcal{P}$ is false and false if $\mathcal{P}$ is true,

$$\mathcal{P}_\text{A} \Rightarrow \mathcal{P}_\text{B} \equiv \mathcal{P}_\text{B} \Leftarrow \mathcal{P}_\text{A} \equiv \text{"from } \mathcal{P}_\text{A} \text{ follows } \mathcal{P}_\text{B}\text{"} \equiv \text{"}\mathcal{P}_\text{A} \text{ entails } \mathcal{P}_\text{B}\text{"} \equiv \text{"}\mathcal{P}_\text{A} \text{ show } \mathcal{P}_\text{B}\text{"} \equiv$$
$$\text{"}\mathcal{P}_\text{A} \text{ gives rise to } \mathcal{P}_\text{B}\text{"} \equiv \text{"}\mathcal{P}_\text{A} \text{ implies } \mathcal{P}_\text{B}\text{"} \equiv \text{"}\mathcal{P}_\text{B} \text{ is due to } \mathcal{P}_\text{A}\text{"} \equiv \text{"}\mathcal{P}_\text{B} \text{ is obtainable from } \mathcal{P}_\text{A}\text{"}$$

$$\{\mathcal{P}_\text{A} \Rightarrow \mathcal{P}_\text{B}\} \bigwedge \{\mathcal{P}_\text{B} \Leftarrow \mathcal{P}_\text{A}\} \equiv \{\mathcal{P}_\text{A} \equiv \mathcal{P}_\text{B}\}$$

$$\{\mathcal{P}_\text{A} \parallel \mathcal{P}_\text{B}\} \equiv \text{"}\mathcal{P}_\text{A} \text{ subject to condition } \mathcal{P}_\text{B}\text{"} \equiv \text{"}\mathcal{P}_\text{A} \text{ of which } \mathcal{P}_\text{B}\text{"} \equiv \text{"}\mathcal{P}_\text{A} \text{ where } \mathcal{P}_\text{B}\text{"}$$

$$\text{Æ}\langle \text{A} /\!\!/ \text{B} /\!\!/ \text{C}\rangle \equiv \text{"the being A a specification of B of which C"}$$

where " $/\!\!/$ C" may be absent causing so the absence of "of which C".

Also it is understood IPM $\equiv$ "the first member of" and

$$\Big\{\text{from: } \text{A}_1; \text{A}_2; \ldots; \text{A}_{\hat{i}}; \text{ follows } \text{B}_0 \diamond_1 \text{B}_1 \diamond_2 \text{B}_2 \cdots \diamond_{\hat{i}} \text{B}_{\hat{i}} \diamond_{\hat{i}+1} \text{B}_{\hat{i}+1} \cdots \diamond_{\hat{i}+\hat{j}} \text{B}_{\hat{i}+\hat{j}}\Big\} \equiv$$
$$\Big\{\text{A}_1 \Rightarrow \{\text{B}_0 \diamond_1 \text{B}_1\}; \text{A}_2 \Rightarrow \{\text{B}_1 \diamond_2 \text{B}_2\}; \ldots; \text{A}_{\hat{i}} \Rightarrow \{\text{B}_{\hat{i}-1} \diamond_{\hat{i}} \text{B}_{\hat{i}}\}\Big\}$$

where: each of $\{\diamond_1, \diamond_2, \ldots, \diamond_{\hat{i}+\hat{j}}\}$ is a relational symbol, as for example one of $\{\equiv,\not\equiv,\neq,\Rightarrow\}$; $\{\diamond_{\hat{i}+1}\text{B}_{\hat{i}+1} \cdots \diamond_{\hat{i}+\hat{j}} \text{B}_{\hat{i}+\hat{j}}\}$ may be absent and if is present the validity of its presence is considered evident; each of $\{\text{A}_1, \text{A}_2, \ldots, \text{A}_{\hat{i}}\}$ is replaced by symbol "þ" when is considered evident the validity of the corresponding element of $\{\{\text{B}_0\diamond_1\text{B}_1\}, \{\text{B}_1\diamond_2\text{B}_2\}, \ldots, \{\text{B}_{\hat{i}-1}\diamond_{\hat{i}}\text{B}_{\hat{i}}\}\}$.

A $\{\text{A}_h; h = \check{h}, \hat{h}\}$ is a sequence, and then also a set, of $\hat{h} - \check{h} + 1$ elements. Is implicit $\{h = \check{h}, \hat{h}\} \equiv \{h; h = \check{h}, \hat{h}\}$. A $\{\text{A} /\!\!/ \mathcal{P}\}$ is the set of all the different specifications of A contextually possible when there is the condition $\mathcal{P}$.

A bijection, i.e. a "one-to-one (injective) and onto (surjective)" correspondence, between two sets $\underline{\text{A}}$ and $\underline{\text{B}}$ of which $\underline{\text{A}} \equiv \{\text{A}_h; h = 1, \hat{h}\}$ and $\underline{\text{B}} \equiv \{\text{B}_k; k = 1, \hat{h}\}$, is a set of $\hat{h}$ pairs indicated $\underline{\text{A}} \Leftrightarrow \underline{\text{B}}$ and defined by an $\underline{\text{A}} \Leftrightarrow \underline{\text{B}} \equiv \{\text{A}_h, \text{B}_{\text{K}\langle h\rangle}; h = 1, \hat{h}\}$ of which $\{\text{K}_h; h = 1, \hat{h}\} = \{k = 1, \hat{h}\}$.

A $[\text{A}_{hk}; h = 1, \hat{h}; k = 1, \hat{k}]$ is the matrix that has $\hat{h}$ rows, $\hat{k}$ columns and A_{hk} as element of h-th row and k-th column. A $[\text{B}_k; k = 1, \hat{k}]$ is a column vector i.e. a matrix which has $\hat{k}$ rows and one column.

It's called $\mathfrak{R}\langle\text{G}\rangle$ the set of every different value that can have the quantity G; and $\mathfrak{R}\langle\underline{\text{G}}\rangle$, of which $\underline{\text{G}} \equiv \{\text{G}_n; n = 1, \hat{n}\}$, the set of every different $\hat{n}$-tuple of values that can be respectively assumed by the quantities $\underline{\text{G}}$. A $\hat{n}$-tuple like this is also called point of $\mathfrak{R}\langle\underline{\text{G}}\rangle$.

It is placed $\underline{\text{x}} \equiv \{\text{x}_n; n = 1, \hat{n}\}$. Every analytic function $f(\underline{\text{x}})$ is understood continuous in its definition domain. A $f(\underline{\text{x}}) \in C^{\hat{m}}(\underline{\mathcal{R}})$ is equivalent to saying that $f(\underline{\text{x}})$ is of class $C^{\hat{m}}$ in $\underline{\mathcal{R}}$ and indicates that every its mixed partial derivative of order less or equal to $\hat{m}$ is continuous at every $\underline{\text{x}}$ of which $\underline{\text{x}} \in \underline{\mathcal{R}}$. A value of a $f(\underline{\text{x}})$ is said local when is understood in a particular point of $\mathfrak{R}\langle\underline{\text{x}}\rangle$.

It is implicit that a $f(\underline{x})$ expresses a variable f in the sense of $f = f(\underline{x})$. Hence

$$\{f(\underline{x}), f(\underline{x})\} \Rightarrow \{\mathfrak{R}\langle f\rangle \supseteq \mathfrak{R}\langle f(\underline{x})\rangle \cup \mathfrak{R}\langle f(\underline{x})\rangle\}$$

When they are not possible misunderstanding, as occurs as long as a variable is expressed by a only function, the function name may be replaced by that of the variable: for example one can write y' instead of $y'(x)$. A $\underline{f} = \underline{f}(\underline{x})$, of which $\underline{f} \equiv \{f_k; k = 1, \hat{k}\}$ or $\underline{f}(\underline{x}) \equiv \{f_k(\underline{x}); k = 1, \hat{k}\}$, is equivalent to $\{f_k = f_k(\underline{x}); k = 1, \hat{k}\}$.

2 Two models for approximate a function: the interpolating polynomial and the cubic spline.

Are considered the interpolating polynomial and the cubic spline between the models (for which reference is made in [2], [3], [4], [5], [6], [7], [8]) for approximating a $\mathsf{y}(\mathsf{x})$ of which $\mathfrak{R}\langle \mathsf{x}\rangle \equiv [\mathsf{x}_1, \mathsf{x}_{\hat{\mathrm{p}}}]$.

Both these models are interpolating i.e. they approximate $\mathsf{y}(\mathsf{x})$ on the basis of knowledge of interpolation points $\underline{\mathsf{x}} \leftrightsquigarrow \underline{\mathsf{y}}$ of which

$$\underline{\mathsf{x}} \leftrightsquigarrow \underline{\mathsf{y}} \equiv \{\mathsf{x}_p, \mathsf{y}_p; p = 1, \hat{\mathrm{p}}\} \qquad \underline{\mathsf{x}} \equiv \{\mathsf{x}_p; p = 1, \hat{\mathrm{p}}\} \qquad \underline{\mathsf{y}} \equiv \{\mathsf{y}_p; p = 1, \hat{\mathrm{p}}\}$$

$$\mathsf{y}_p = \mathsf{y}(\mathsf{x}_p) \qquad \{\mathsf{x}_{p-1} < \mathsf{x}_p; p = 2, \hat{\mathrm{p}}\}$$

and they assume the $\underline{\mathsf{y}}$ values at the respective $\underline{\mathsf{x}}$.

2.1 The interpolating polynomial

The interpolating polynomial $\mathcal{P}_{\mathrm{I}}(\mathsf{x})$, which interpolates the $\underline{\mathsf{x}} \leftrightsquigarrow \underline{\mathsf{y}}$, is the polynomial, of degree at most $\hat{\mathrm{p}} - 1$, expressed by

$$\mathcal{P}_{\mathrm{I}}(\mathsf{x}) \equiv \sum_{p=1}^{\hat{\mathrm{p}}} \Gamma_p \mathsf{x}^{p-1} = \sum_{p=1}^{\hat{\mathrm{p}}} \mathsf{x}^{p-1} \sum_{\mathrm{p}=1}^{\hat{\mathrm{p}}} \mathit{x}_{p\mathrm{p}} \mathsf{y}_{\mathrm{p}} = \sum_{\mathrm{p}=1}^{\hat{\mathrm{p}}} \mathsf{y}_{\mathrm{p}} \lambda_{\mathrm{PIp}}(\mathsf{x}) \tag{1}$$

of which

$$\{\Gamma_{\mathrm{p}}; \mathrm{p} = 1, \hat{\mathrm{p}}\} \equiv \underline{\Gamma} = \underline{\underline{\mathsf{x}}}^{-1} \cdot \underline{\mathsf{y}} \qquad \underline{\underline{\mathsf{x}}} \equiv [\mathsf{x}_{\mathrm{p}}{}^{p-1}; \mathrm{p} = 1, \hat{\mathrm{p}}; p = 1, \hat{\mathrm{p}}]$$

$$\underline{\underline{\mathsf{x}}}^{-1} \equiv [\mathit{x}_{p\mathrm{p}}; \mathrm{p} = 1, \hat{\mathrm{p}}; p = 1, \hat{\mathrm{p}}] \qquad \lambda_{\mathrm{PIp}}(\mathsf{x}) \equiv \sum_{p=1}^{\hat{\mathrm{p}}} \mathit{x}_{p\mathrm{p}} \mathsf{x}^{p-1} \qquad \lambda'_{\mathrm{PIp}}(\mathsf{x}) \equiv \sum_{p=1}^{\hat{\mathrm{p}}} (p-1) \mathit{x}_{p\mathrm{p}} \mathsf{x}^{p-2}$$

The linear system $\underline{\underline{\mathsf{x}}} \cdot \underline{\Gamma} = \underline{\mathsf{y}}$, that defines the coefficients $\underline{\Gamma}$ of $\mathcal{P}_{\mathrm{I}}(\mathsf{x})$, is equivalent to express $\{\mathcal{P}_{\mathrm{I}}(\mathsf{x}_{\mathrm{p}}) = \mathsf{y}_{\mathrm{p}}; \mathrm{p} = 1, \hat{\mathrm{p}}\}$ using the second member of (1). The (1) has at $\mathsf{x} = 0$ a singular point which however, on the basis of (2.4.2.10) of [1], it is understood eliminated by

$$\mathcal{P}_{\mathrm{I}}(0) = \lim_{\mathsf{x}\to 0}(\Gamma_1 \mathsf{x}^0 + \textstyle\sum_{p=2}^{\hat{\mathrm{p}}} \Gamma_p \mathsf{x}^{p-1}) = \Gamma_1$$

which follows from $\lim_{\mathsf{x}\to 0}\mathsf{x}^0 = 1$ (which also eliminates a possible indeterminate form 0^0 that could happen in $\underline{\underline{\mathsf{X}}}$).

The $\underline{\underline{\mathsf{X}}}$, as well as its transpose $\underline{\underline{\mathsf{X}}}^\intercal$, is a matrix type said of Vandermonde, of which

$$\det\underline{\underline{\mathsf{X}}} = \prod_{\mathrm{p}=1}^{\hat{\mathrm{p}}-1}\prod_{p=\mathrm{p}+1}^{\hat{\mathrm{p}}} \mathsf{x}_p - \mathsf{x}_\mathrm{p} \neq 0$$

that entail the existence of $\underline{\underline{\mathsf{X}}}^{-1}$ and so the existence and uniqueness of $\mathcal{P}_\mathrm{I}(\mathsf{x})$.

The $\underline{\underline{\boldsymbol{\delta}}}_{\hat{n}}$, of which $\underline{\underline{\boldsymbol{\delta}}}_{\hat{n}} \equiv [\delta_{\mathrm{n}n}; \mathrm{n} = 1, \hat{n}; n = 1, \hat{n}]$, $\{\delta_{\mathrm{n}n} = 0; \forall\mathrm{n} \neq n\}$ and $\{\delta_{\mathrm{n}n} = 1; \forall\mathrm{n} = n\}$, is the identity matrix (also said unit matrix). For $\underline{\underline{\mathrm{A}}} \equiv [\mathrm{A}_{mn}; m = 1, \hat{m}; n = 1, \hat{n}]$, $|\underline{\underline{\mathrm{A}}}| \equiv [|\mathrm{A}_{mn}|; m = 1, \hat{m}; n = 1, \hat{n}]$ defines $|\underline{\underline{\mathrm{A}}}|$ as the absolute value of $\underline{\underline{\mathrm{A}}}$. In the case $\hat{m} = \hat{n}$, the numerical error of $\underline{\underline{\mathrm{A}}}$ inversion can be measured by the maximum value in the matrix $|\underline{\underline{\mathrm{A}}} \cdot \underline{\underline{\mathrm{A}}}^{-1} - \underline{\underline{\boldsymbol{\delta}}}_{\hat{n}}|$.

The $\underline{\underline{\mathsf{X}}}$ is typically ill-conditioned since the calculation of $\underline{\underline{\mathsf{X}}}^{-1}$ can induce that the maximum in $|\underline{\underline{\mathsf{X}}} \cdot \underline{\underline{\mathsf{X}}}^{-1} - \underline{\underline{\boldsymbol{\delta}}}_{\hat{p}}|$ is important, even if $\hat{\mathrm{p}}$ is not large and the numbers are represented by a long sequence of digits. This inconvenience also relates to the calculation of $\underline{\Gamma}$ as solution of $\underline{\underline{\mathsf{X}}} \cdot \underline{\Gamma} = \underline{\mathsf{y}}$.

The same $\mathcal{P}_\mathrm{I}(\mathsf{x})$ expressed by (1) has also the Lagrange form

$$\mathcal{P}_\mathrm{I}(\mathsf{x}) = \sum_{\mathrm{p}=1}^{\hat{\mathrm{p}}} \frac{Д_\mathrm{p}(\mathsf{x})}{Д_\mathrm{p}(\mathsf{x}_\mathrm{p})}\mathsf{y}_\mathrm{p} = Д(\mathsf{x}) \sum_{\mathrm{p}=1}^{\hat{\mathrm{p}}} \frac{\mathsf{y}_\mathrm{p}}{(\mathsf{x} - \mathsf{x}_\mathrm{p})\, Д_\mathrm{p}(\mathsf{x}_\mathrm{p})} \tag{2}$$

of which

$$Д_\mathrm{p}(\mathsf{x}) \equiv \prod_{p=1}^{\hat{\mathrm{p}}} \delta_{p\mathrm{p}} + (\mathsf{x} - \mathsf{x}_p)(1 - \delta_{p\mathrm{p}}) \qquad Д(\mathsf{x}) \equiv \prod_{p=1}^{\hat{\mathrm{p}}} (\mathsf{x} - \mathsf{x}_p)$$

and showing $\lambda_{\mathrm{PIp}}(\mathsf{x}) = Д_\mathrm{p}(\mathsf{x})/Д_\mathrm{p}(\mathsf{x}_\mathrm{p})$,

$$\lambda'_{\mathrm{PIp}}(\mathsf{x}) = Д_\mathrm{p}^{-1}(\mathsf{x}_\mathrm{p}) \sum_{a=1}^{\hat{\mathrm{p}}} (1 - \delta_{a\mathrm{p}}) \prod_{p=1}^{\hat{\mathrm{p}}} \delta_{pa} + (1 - \delta_{pa})\big(\delta_{p\mathrm{p}} + (1 - \delta_{p\mathrm{p}})(\mathsf{x} - \mathsf{x}_p)\big)$$

The (2) is advantageously used in place of (1), since, by not requiring the calculation of $\underline{\underline{\mathsf{X}}}^{-1}$ nor the solution of $\underline{\underline{\mathsf{X}}} \cdot \underline{\Gamma} = \underline{\mathsf{y}}$, implies fewer burdens and numerical errors.

Regardless of the particular form that expresses $\mathcal{P}_\mathrm{I}(\mathsf{x})$, which affects only its numerical aspect, for its analytical error $\mathcal{E}_\mathrm{PI}(\mathsf{x})$ is had

$$\mathcal{E}_\mathrm{PI}(\mathsf{x}) \equiv \mathsf{y}(\mathsf{x}) - \mathcal{P}_\mathrm{I}(\mathsf{x}) = \frac{\mathsf{y}^{(\hat{\mathrm{p}})}(\xi_1(\mathsf{x}))}{\hat{\mathrm{p}}!} Д(\mathsf{x}) \tag{3}$$

$$\mathcal{E}_\mathrm{PI}{}^{(k)}(\mathsf{x}) \equiv \mathsf{y}^{(k)}(\mathsf{x}) - \mathcal{P}_\mathrm{I}{}^{(k)}(\mathsf{x}) = \frac{\mathsf{y}^{(\hat{\mathrm{p}})}(\xi_2(\mathsf{x}))}{(\hat{\mathrm{p}} - k)!} \prod_{p=1}^{\hat{\mathrm{p}}-k} \mathsf{x} - \zeta_{kp} \tag{4}$$

of which

$$1 \le k \le \hat{\mathrm{p}} - 1 \qquad \{\xi_i(\mathsf{x}) \in (\mathsf{x}_1, \mathsf{x}_{\hat{\mathrm{p}}}); i = 1, 2\}$$

$$\{\zeta_{kp} \in (\mathsf{x}_p, \mathsf{x}_{p+k}); p = 1, \hat{\mathrm{p}} - k\} \qquad \{\mathsf{y}(\mathsf{x}) \in C^{\hat{p}}(\mathfrak{R}_{\mathsf{x}})\} \Rightarrow \{(3), (4)\}$$

The (3) and (4) imply the respective

$$|\mathcal{E}_{\text{PI}}(\mathsf{x})| = \frac{|\mathsf{y}^{(\hat{\mathrm{p}})}(\xi_1(\mathsf{x}))|}{\hat{\mathrm{p}}!} |Д(\mathsf{x})| \le \frac{\Phi\langle \mathsf{y}^{(\hat{\mathrm{p}})}\rangle}{\hat{\mathrm{p}}!} \prod_{p=1}^{\hat{\mathrm{p}}} |\mathsf{x} - \mathsf{x}_p|$$

$$\left|\mathcal{E}_{\text{PI}}{}^{(k)}(\mathsf{x})\right| = \frac{|\mathsf{y}^{(\hat{\mathrm{p}})}(\xi_2(\mathsf{x}))|}{(\hat{\mathrm{p}} - k)!} \prod_{p=1}^{\hat{\mathrm{p}}-k} |\mathsf{x} - \zeta_{kp}| \le \frac{\Phi\langle \mathsf{y}^{(\hat{\mathrm{p}})}\rangle}{(\hat{\mathrm{p}} - k)!} \prod_{p=1}^{\hat{\mathrm{p}}-k} \hat{\Delta}_p$$

of which $\Phi\langle \mathsf{y}^{(p)}\rangle \equiv \max\{|\mathsf{y}^{(p)}(\mathsf{x})| /\!\!/ \mathsf{x} \in (\mathsf{x}_1, \mathsf{x}_{\hat{\mathrm{p}}})\}$, $\hat{\Delta}_p \equiv \max\{|\mathsf{x} - \mathsf{x}_p|, |\mathsf{x} - \mathsf{x}_{p+k}|\}$.

The $\mathcal{P}_{\text{I}}(\mathsf{x})$ has as inconveniences oscillatory behavior and eventuality that this increases with $\hat{\mathrm{p}}$. Indeed, also sensibly expecting for most of cases that, with increasing the $\hat{\mathrm{p}}$ of $\underline{\mathsf{x}}$ equidistant, the respective interpolating polynomials converge neatly to $\mathsf{y}(\mathsf{x})$, nevertheless this convergence is not always surely attainable, having however in this regard the improvement of replace the $\underline{\mathsf{x}}$ equidistant with the Chebyshev points defined by

$$\{\mathsf{x}_p = \frac{\mathsf{x}_{\hat{\mathrm{p}}} + \mathsf{x}_1}{2} + \frac{\mathsf{x}_{\hat{\mathrm{p}}} - \mathsf{x}_1}{2} \cos(\frac{\hat{\mathrm{p}} - p}{\hat{\mathrm{p}} - 1} \pi); p = 1, \hat{\mathrm{p}}\}$$

2.2 The cubic spline

The natural cubic spline, that interpolates the $\underline{\mathsf{x}} \leftrightsquigarrow \underline{\mathsf{y}}$, is the piecewise polynomial function made up by the $\hat{\mathrm{p}} - 1$ polynomials $\{\mathcal{S}_p(\mathsf{x}); p = \overline{1}, \hat{\mathrm{p}} - 1\}$, in the sense of

$$\mathcal{S}(\mathsf{x}) = \mathcal{S}_p(\mathsf{x}) \equiv \{\mathcal{S}_p(\mathsf{x}) \parallel \mathsf{x} \in \mathfrak{I}_p\} \tag{5}$$

of which $\mathfrak{I}_p \equiv [\mathsf{x}_p, \mathsf{x}_{p+1}]$.

Each of $\{\mathcal{S}_p(\mathsf{x}); p = 1, \hat{\mathrm{p}} - 1\}$ is defined in the corresponding $\mathfrak{I}_p$ and has degree at most 3. The linear system of $4(\hat{\mathrm{p}} - 1)$ equations

$$\{\mathcal{S}_p(\mathsf{x}_p) = \mathsf{y}_p, \mathcal{S}_p(\mathsf{x}_{p+1}) = \mathsf{y}_{p+1}; p = 1, \hat{\mathrm{p}} - 1\} \tag{6}$$

$$\{\mathcal{S}'_{p-1}(\mathsf{x}_p) = \mathcal{S}'_p(\mathsf{x}_p); p = 2, \hat{\mathrm{p}} - 1\} \tag{7}$$

$$\{\mathcal{S}''_{p-1}(\mathsf{x}_p) = \mathcal{S}''_p(\mathsf{x}_p); p = 2, \hat{\mathrm{p}} - 1\} \tag{8}$$

$$\{\mathcal{S}''_1(\mathsf{x}_1) = 0, \mathcal{S}''_{\hat{\mathrm{p}}-1}(\mathsf{x}_{\hat{\mathrm{p}}}) = 0\} \tag{9}$$

is solvable in just as many unknowns constituted by the coefficients of these $\hat{\mathrm{p}} - 1$ polynomials.

The numerical burden of knowing the $\{\mathcal{S}_p(\mathsf{x}); p = 1, \hat{\mathrm{p}} - 1\}$ by solving this system with an algorithm of general applicability, as that of Gauss with the strategy of the “maximum pivot” referred in [7] and [9], can be reduced by transforming such system as follows.

Are established $\{\mathrm{U}_p \equiv \mathcal{S}''_p(\mathsf{x}_p); p = 1, \hat{\mathrm{p}} - 1\}$ and $\mathrm{U}_{\hat{\mathrm{p}}} \equiv \mathcal{S}''_{\hat{\mathrm{p}}-1}(\mathsf{x}_{\hat{\mathrm{p}}})$, that allow both replace (8) with the understand $\{\mathrm{U}_p \equiv \mathcal{S}''_{p-1}(\mathsf{x}_p) = \mathcal{S}''_p(\mathsf{x}_p); p = 2, \hat{\mathrm{p}} - 1\}$ and write (9) in the form $\mathrm{U}_1 = \mathrm{U}_{\hat{\mathrm{p}}} = 0$.

The definition of $\mathcal{S}_p(\mathsf{x})$ as a polynomial of degree at most 3 implies that $\mathcal{S}''_p(\mathsf{x})$ is a polynomial of degree at most 1. This deduction, $\mathrm{U}_p = \mathcal{S}''_p(\mathsf{x}_p)$ and $\mathrm{U}_{p+1} = \mathcal{S}''_p(\mathsf{x}_{p+1})$ entail

$$Æ\langle \mathcal{S}''_p(\mathsf{x}), \{\mathsf{x}_a, \mathrm{U}_a; a = p, p+1\} \mathbin{/\!/} \mathcal{P}_{\mathrm{I}}(\mathsf{x}), \underline{\mathsf{x}} \twoheadleftarrow\!\!\!\twoheadrightarrow \underline{\mathsf{y}} \mathbin{/\!/} (2)\rangle$$

that gives rise to

$$\mathcal{S}''_p(\mathsf{x}) = \frac{\mathrm{U}_{p+1}(\mathsf{x} - \mathsf{x}_p) - \mathrm{U}_p(\mathsf{x} - \mathsf{x}_{p+1})}{\Delta_p} \tag{10}$$

of which $\Delta_p = \mathsf{x}_{p+1} - \mathsf{x}_p$ and that, on the basis of (5), show $\mathcal{S}(\mathsf{x}) \in C^2(\mathfrak{R}_{\mathsf{x}})$.

From: $\mathcal{S}'_p(\mathsf{x}) - \mathrm{V}_p = \int \mathcal{S}''_p(\mathsf{x})\,\mathrm{d}\mathsf{x}$, due to $\{F(\mathrm{x}) \equiv \int f(\mathrm{x})\,\mathrm{dx}\} \equiv \{F'(\mathrm{x}) = f(\mathrm{x})\}$; (10); follows

$$\mathcal{S}'_p(\mathsf{x}) = \int \mathcal{S}''_p(\mathsf{x})\,\mathrm{d}\mathsf{x} + \mathrm{V}_p = \int \frac{\mathrm{U}_{p+1}(\mathsf{x} - \mathsf{x}_p) - \mathrm{U}_p(\mathsf{x} - \mathsf{x}_{p+1})}{\Delta_p}\,\mathrm{d}\mathsf{x} + \mathrm{V}_p =$$
$$\frac{\mathrm{U}_{p+1}(\mathsf{x} - \mathsf{x}_p)^2 - \mathrm{U}_p(\mathsf{x} - \mathsf{x}_{p+1})^2}{2\Delta_p} + \mathrm{V}_p \tag{11}$$

From: $\mathcal{S}_p(\mathsf{x}) - \mathrm{W}_p = \int \mathcal{S}'_p(\mathsf{x})\,\mathrm{d}\mathsf{x}$; (11); follows

$$\mathcal{S}_p(\mathsf{x}) = \int \mathcal{S}'_p(\mathsf{x})\,\mathrm{d}\mathsf{x} + \mathrm{W}_p = \int \frac{\mathrm{U}_{p+1}(\mathsf{x} - \mathsf{x}_p)^2 - \mathrm{U}_p(\mathsf{x} - \mathsf{x}_{p+1})^2}{2\Delta_p}\,\mathrm{d}\mathsf{x} + \int \mathrm{V}_p\,\mathrm{d}\mathsf{x} + \mathrm{W}_p =$$
$$\frac{\mathrm{U}_{p+1}(\mathsf{x} - \mathsf{x}_p)^3 - \mathrm{U}_p(\mathsf{x} - \mathsf{x}_{p+1})^3}{6\Delta_p} + \mathrm{V}_p(\mathsf{x} - \mathsf{x}_p) + \mathrm{W}_p \tag{12}$$

This deduction allows to write (6) as

$$\{\mathrm{W}_p = \mathsf{y}_p - \frac{\Delta_p^2 \mathrm{U}_p}{6}, \mathrm{V}_p = \frac{\mathsf{y}_{p+1} - \mathsf{y}_p}{\Delta_p} - \frac{\mathrm{U}_{p+1} - \mathrm{U}_p}{6}\Delta_p; p = 1, \hat{\mathrm{p}} - 1\} \tag{13}$$

The (11) and expression of V_p in (13) allow to write (7) as

$$\{\frac{\Delta_p \mathrm{U}_p}{6} + \frac{\Delta_p + \Delta_{p+1}}{3}\mathrm{U}_{p+1} + \frac{\Delta_{p+1}\mathrm{U}_{p+2}}{6} = \mathsf{Y}_p; p = 1, \hat{\mathrm{p}} - 2\}$$

of which

$$\mathsf{Y}_p = \frac{\mathsf{y}_{p+2} - \mathsf{y}_{p+1}}{\Delta_{p+1}} - \frac{\mathsf{y}_{p+1} - \mathsf{y}_p}{\Delta_p}$$

and hence, using the matrix notation and introducing (9) in the form $\mathrm{U}_1 = \mathrm{U}_{\hat{\mathrm{p}}} = 0$, as the linear system $\underline{\underline{\mathrm{S}}} \cdot \underline{\mathrm{U}} = \underline{\mathsf{Y}}$ of $\hat{\mathrm{p}} - 2$ equations in the just as many unknowns $\underline{\mathrm{U}}$, defined by

$$\underline{\mathrm{U}} \equiv \{\mathrm{U}_p; p = 2, \hat{\mathrm{p}} - 1\} \quad \underline{\mathsf{Y}} \equiv \{\mathsf{Y}_p; p = 1, \hat{\mathrm{p}} - 2\} \quad \underline{\underline{\mathrm{S}}} \equiv [\mathrm{S}_{\mathrm{p}p}; \mathrm{p} = 1, \hat{\mathrm{p}} - 2; p = 1, \hat{\mathrm{p}} - 2]$$

with the elements of $\underline{\underline{S}}$ all null except

$$S_{11} = \frac{\Delta_1 + \Delta_2}{3} \qquad S_{12} = \frac{\Delta_2}{6} \qquad S_{\hat{p}-2,\hat{p}-3} = \frac{\Delta_{\hat{p}-2}}{6} \qquad S_{\hat{p}-2,\hat{p}-2} = \frac{\Delta_{\hat{p}-2} + \Delta_{\hat{p}-1}}{3}$$

$$\{S_{p,p-1} = \frac{\Delta_p}{6},\ S_{pp} = \frac{\Delta_p + \Delta_{p+1}}{3},\ S_{p,p+1} = \frac{\Delta_{p+1}}{6}; p = 2, \hat{p} - 3\}$$

and being such system equivalent to $\underline{U} = \underline{\underline{S}}^{-1} \cdot \underline{Y}$ of which $\underline{\underline{S}}^{-1} \equiv [\mathcal{S}_{pp}; p = 1, \hat{p}-2; p = 1, \hat{p}-2]$ and hence to

$$\{U_{p+1} = \sum_{p=1}^{\hat{p}-2} \mathcal{S}_{pp} Y_p; p = 1, \hat{p} - 2\}$$

The system constituted by (6), (7), (8) and (9) was transformed in that constituted by (13), $\underline{U} = \underline{\underline{S}}^{-1} \cdot \underline{Y}$ and $U_1 = U_{\hat{p}} = 0$. Therefore the introduction of these into (12) and (11) makes known the expressions of $\{\mathcal{S}_p(x), \mathcal{S}'_p(x); p = 1, \hat{p} - 1\}$, having in particular

$$\{\mathcal{S}'(x_p) = \mathcal{S}'_p(x_p) = \sum_{p=1}^{\hat{p}} \lambda_{pp} y_p; \forall p < \hat{p}\} \qquad \mathcal{S}'(x_{\hat{p}}) = \mathcal{S}'_{\hat{p}-1}(x_{\hat{p}}) = \sum_{p=1}^{\hat{p}} \vartheta_p y_p \tag{14}$$

of which

$$\lambda_{pp} \equiv \frac{\delta_{p,p+1}\tilde{\delta}_{p1} - \delta_{pp}\tilde{\delta}_{p\hat{p}}}{\Delta_p} - (\frac{\tilde{\delta}_{p1}\tilde{\delta}_{p\hat{p}}\tilde{K}_{pp}}{3} + \frac{\tilde{\delta}_{p+1,\hat{p}}\tilde{K}_{p+1,p}}{6})\Delta_p$$

$$\vartheta_p \equiv \frac{\delta_{p\hat{p}} - \delta_{p,\hat{p}-1}}{\Delta_{\hat{p}-1}} + \frac{\tilde{\delta}_{\hat{p}-1,1}\tilde{K}_{\hat{p}-1,p}}{6}\Delta_{\hat{p}-1}$$

$$\tilde{K}_{pp} \equiv \frac{\tilde{\delta}_{p,\hat{p}-1}\tilde{\delta}_{p\hat{p}}\mathcal{S}_{p-1,p}}{\Delta_p} - \tilde{\delta}_{p\hat{p}}\tilde{\delta}_{p1}\mathcal{S}_{p-1,p-1}(\Delta_{p-1}^{-1} + \Delta_p^{-1}) + \frac{\tilde{\delta}_{p1}\tilde{\delta}_{p2}\mathcal{S}_{p-1,p-2}}{\Delta_{p-1}}$$

where is understood $\tilde{\delta}_{ab} \equiv 1 - \delta_{ab}$ whence $\{\tilde{\delta}_{ab} = 1; \forall a \neq b\}$, $\{\tilde{\delta}_{ab} = 0; \forall a = b\}$, and is considered null every addend where it appears at least a factor null.

The complete cubic spline $\mathcal{S}_C(x)$ differs from $\mathcal{S}(x)$ only for the substitution of (9) with the assignment of known values to $\mathcal{S}'_C(x_1)$ and $\mathcal{S}'_C(x_{\hat{p}})$. The periodic cubic spline $\mathcal{S}_P(x)$ differs from $\mathcal{S}(x)$ only for the substitution of (9) with $\mathcal{S}'_P(x_1) = \mathcal{S}'_P(x_{\hat{p}})$ and $\mathcal{S}''_P(x_1) = \mathcal{S}''_P(x_{\hat{p}})$ when $y(x_1) = y(x_{\hat{p}})$.

For $\mathcal{S}_C(x)$ we have (in [5]) the

$$\begin{aligned} |\mathcal{E}_{SC}(x)| = |y(x) - \mathcal{S}_C(x)| \le \frac{7}{8}\hat{\Phi}\langle y^{(4)}\rangle\frac{\hat{\Delta}^5}{\check{\Delta}} \\ |\mathcal{E}'_{SC}(x)| = |y'(x) - \mathcal{S}'_C(x)| \le \frac{7}{4}\hat{\Phi}\langle y^{(4)}\rangle\frac{\hat{\Delta}^4}{\check{\Delta}} \end{aligned} \tag{15}$$

of which

$$\hat{\Phi}\langle y^{(p)}\rangle \equiv \max\{|y^{(p)}(x)| /\!/ x \in \mathfrak{R}_x\} \qquad \hat{\Delta} \equiv \max\{\Delta_p; p = 1, \hat{p} - 1\}$$

$$\check{\Delta} \equiv \min\{\Delta_p; p = 1, \hat{p} - 1\} \qquad \{y(x) \in C^4(\mathfrak{R}_x)\} \Rightarrow (15)$$

Other bounds for the errors of the cubic spline are in [2], [3], [4], [8].

With the increase of $\hat{\mathrm{p}}$, each subsequent $\mathcal{S}_{\mathrm{C}}(\mathsf{x})$ and its derivatives up to second order converges to $\mathsf{y}(\mathsf{x})$ and its corresponding derivative, with the only condition that $\hat{\Delta}/\check{\Delta}$ remains limited, and having in the case of $\underline{\mathsf{x}}$ equidistant the most rapid convergence indicated by

$$|\mathcal{E}_{\mathrm{SC}}(\mathsf{x})| \leq \frac{7}{8}\hat{\Delta}^4\hat{\Phi}\langle \mathsf{y}^{(4)}\rangle$$

In the limit as $\hat{\mathrm{p}} \to \infty$, i.e. as $\hat{\mathrm{p}}$ approaches ∞, consistent with the said condition, the (15) are valid also for $\mathcal{S}(\mathsf{x})$ inasmuch this coincides with $\mathcal{S}_{\mathrm{C}}(\mathsf{x})$.

The inconveniences of $\mathcal{P}_{\mathrm{I}}(\mathsf{x})$, said in section 2.1, are resolved in an excellent way by $\mathcal{S}(\mathsf{x})$, because the oscillations of this are minimal among those of all the different functions of class C^2 in $\mathfrak{R}_{\mathsf{x}}$ which interpolate the $\underline{\mathsf{x}} \leftrightarrow \underline{\mathsf{y}}$, and for the said properties of convergence to the exact value with the increase of $\hat{\mathrm{p}}$.

2.3 New bounds for the errors of a cubic spline

It is called $\tilde{\mathcal{S}}(\mathsf{x})$ a cubic spline which differs from $\mathcal{S}(\mathsf{x})$ only for the replacement of (9) with another two equations, thus having $Æ\langle \mathcal{S} \mathbin{/\!/} \tilde{\mathcal{S}}\rangle$, $Æ\langle \mathcal{S}_{\mathrm{C}} \mathbin{/\!/} \tilde{\mathcal{S}}\rangle$ and $Æ\langle \mathcal{S}_{\mathrm{P}} \mathbin{/\!/} \tilde{\mathcal{S}}\rangle$. Coherently with this, is understood the possibility of replacing $\mathcal{S}$ with $\tilde{\mathcal{S}}(\mathsf{x})$ when treating properties independent from (9).

From: $\mathscr{P}_{g(\underline{x})}$ of which $\mathscr{P}_{g(\underline{x})} \equiv \{g(\underline{x})/|g(\underline{x})| \equiv \boldsymbol{\omega}_{g(\underline{x})}; \forall \underline{x} \in \mathcal{R}_{\underline{x}}\}$ with $\boldsymbol{\omega}_{g(\underline{x})}$ constant; first mean value theorem for integration said in section 2.4.4 of [1]; follows

$$\int_{\mathcal{R}_{\underline{x}}} f(\underline{x})g(\underline{x})\,\mathrm{d}\underline{x} = \boldsymbol{\omega}_{g(\underline{x})}\int_{\mathcal{R}_{\underline{x}}} f(\underline{x})|g(\underline{x})|\,\mathrm{d}\underline{x} = f(\underline{x})\int_{\mathcal{R}_{\underline{x}}} g(\underline{x})\,\mathrm{d}\underline{x} \tag{16}$$

of which $\mathscr{P}_{g(\underline{x})} \Rightarrow$ (16), $\underline{x} \in \mathcal{R}_{\underline{x}}$, and that gives rise to $\int_{\mathcal{R}_{\underline{x}}} f(\underline{x})\,\mathrm{d}\underline{x} = f(\underline{x})\int_{\mathcal{R}_{\underline{x}}}\mathrm{d}\underline{x}$ if $g(\underline{x}) = 1$.

The (10) and (5) entail

$$\begin{gathered} \mathsf{x} \in (\mathsf{x}_p, \mathsf{x}_{p+1}) \Rightarrow \{\tilde{\mathcal{S}}^{(3)}(\mathsf{x}) = \frac{\mathrm{U}_{p+1} - \mathrm{U}_p}{\Delta_p}\} \\ \tilde{\mathcal{S}}^{(3)}(\mathsf{x}_1) = \frac{\mathrm{U}_2 - \mathrm{U}_1}{\Delta_1} \qquad \tilde{\mathcal{S}}^{(3)}(\mathsf{x}_{\hat{\mathrm{p}}}) = \frac{\mathrm{U}_{\hat{\mathrm{p}}} - \mathrm{U}_{\hat{\mathrm{p}}-1}}{\Delta_{\hat{\mathrm{p}}-1}} \end{gathered} \tag{17}$$

Are considered as implicit

$$\begin{gathered} \mathsf{y}(\mathsf{x}) \in C^4(\mathfrak{R}_{\mathsf{x}}) \qquad \mathcal{E}_p^{(\circ)} \equiv \mathcal{E}^{(\circ)}(\mathsf{x}_p) \qquad \mathcal{E}^{(\circ)}(\mathsf{x}) \equiv \tilde{\mathcal{S}}^{(\circ)}(\mathsf{x}) - \mathsf{y}^{(\circ)}(\mathsf{x}) \\ \mathfrak{R}\langle t_p\rangle \equiv (\mathsf{x}_p, \mathsf{x}_{p+1}) \qquad \acute{\mathrm{\imath}}_p(\mathrm{t}) \equiv \mathrm{t} - \mathsf{x}_p \qquad \grave{\mathrm{\imath}}_p(\mathrm{t}) \equiv \mathrm{t} - \mathsf{x}_{p+1} \qquad \boldsymbol{\omega}\langle \mathrm{G}\rangle \equiv \mathrm{G}/|\mathrm{G}| \end{gathered}$$

The

$$\{Æ\langle \mathsf{x}_p, \mathsf{x}_{p+1} \mathbin{/\!/} \underline{\mathsf{x}} \mathbin{/\!/} (2)\rangle, \mathrm{t} \in \mathfrak{I}_p\} \Rightarrow \{\mathcal{P}_{\mathrm{I}}(\mathrm{t}) = \frac{\acute{\mathrm{\imath}}_p(\mathrm{t})\mathsf{y}_{p+1} - \grave{\mathrm{\imath}}_p(\mathrm{t})\mathsf{y}_p}{\Delta_p}\}$$

$$\{Æ\langle\{\mathsf{x}_p, \mathsf{x}_{p+1}\}, \{\mathcal{E}''_p, \mathcal{E}''_{p+1}\} \mathbin{/\!/} \underline{\mathsf{x}}, \underline{\mathsf{y}} \mathbin{/\!/} (3)\rangle, \mathrm{t} \in \mathfrak{I}_p\} \Rightarrow \{\mathcal{E}''(\mathrm{t}) - \mathcal{P}_{\mathrm{I}}(\mathrm{t}) = \frac{\acute{\mathrm{\imath}}_p(\mathrm{t})\grave{\mathrm{\imath}}_p(\mathrm{t})\mathcal{E}^{(4)}(\xi_{\mathrm{A}}(\mathrm{t}))}{2}\}$$

of which $\{\mathrm{t} \in \mathfrak{I}_p\} \Rightarrow \{\xi_{\mathrm{A}}(\mathrm{t}) \in \mathfrak{R}_{t_p}\}$, and $\{\mathrm{t} \in \mathfrak{R}_{t_p}\} \Rightarrow \{\mathcal{E}^{(4)}(\mathrm{t}) = -\mathsf{y}^{(4)}(\mathrm{t})\}$ due to (17), entail $\mathsf{A}_p(t_p) = 0$ of which

$$\mathsf{A}_p(\mathrm{t}) \equiv \tilde{\mathsf{A}}_p(\mathrm{t}) + \frac{\acute{\mathrm{i}}_p(\mathrm{t})\grave{\mathrm{i}}_p(\mathrm{t})\Delta_p}{2}\mathsf{y}^{(4)}(\xi_{\mathrm{A}}(\mathrm{t})) \qquad \tilde{\mathsf{A}}_p(\mathrm{t}) \equiv \Delta_p\mathcal{E}''(\mathrm{t}) + \grave{\mathrm{i}}_p(\mathrm{t})\mathcal{E}''_p - \acute{\mathrm{i}}_p(\mathrm{t})\mathcal{E}''_{p+1}$$

From: these, distributive property of integration and

$$\int_{\mathcal{R}_{\underline{\mathrm{x}}}} f(\underline{\mathrm{x}})\,\mathrm{d}\underline{\mathrm{x}} = \lim_{\mathcal{R}\to\mathcal{R}_{\underline{\mathrm{x}}}} \int_{\mathcal{R}} f(\underline{\mathrm{x}})\,\mathrm{d}\underline{\mathrm{x}};$$

(16), and constancy of $\boldsymbol{\omega}\langle\acute{\mathrm{i}}_p(\mathrm{t})\,\grave{\mathrm{i}}_p(\mathrm{t})\rangle$ if $\mathrm{t} \in \mathfrak{I}_p$; $\acute{\mathrm{i}}_p(\mathrm{t})\,\grave{\mathrm{i}}_p(\mathrm{t}) = \acute{\mathrm{i}}^2_p(\mathrm{t}) - \Delta_p\acute{\mathrm{i}}_p(\mathrm{t})$ due to $\Delta_p \equiv \acute{\mathrm{i}}_p(\mathrm{t}) - \grave{\mathrm{i}}_p(\mathrm{t})$;

$$\lim_{a\to\mathsf{x}_p^+} \int_a^{t_p} \mathsf{A}_p(\mathrm{t})\,\mathrm{dt} = 0$$

due to $\mathsf{A}_p(t_p) = 0$ and $\mathfrak{R}_{t_p} \equiv (\mathsf{x}_p, \mathsf{x}_{p+1})$; follows

$$\lim_{a\to\mathsf{x}_p^+} \int_a^{t_p} \mathsf{A}_p(\mathrm{t})\,\mathrm{dt} = \int_{\mathsf{x}_p}^{t_p} \tilde{\mathsf{A}}_p(\mathrm{t})\,\mathrm{dt} + \frac{\Delta_p}{2}\int_{\mathsf{x}_p}^{t_p} \acute{\mathrm{i}}_p(\mathrm{t})\,\grave{\mathrm{i}}_p(\mathrm{t})\mathsf{y}^{(4)}(\xi_{\mathrm{A}}(\mathrm{t}))\,\mathrm{dt} =$$
$$\int_{\mathsf{x}_p}^{t_p} \tilde{\mathsf{A}}_p(\mathrm{t})\,\mathrm{dt} + \frac{\Delta_p}{2}\mathsf{y}^{(4)}(\xi_{\mathrm{B}}(t_p))\int_{\mathsf{x}_p}^{t_p} \acute{\mathrm{i}}_p(\mathrm{t})\,\grave{\mathrm{i}}_p(\mathrm{t})\,\mathrm{dt} = \mathsf{B}_p(t_p) = 0 \quad (18)$$

of which $\xi_{\mathrm{B}}(t_p) \in \mathfrak{R}_{t_p}$ and

$$\mathsf{B}_p(\mathrm{t}) \equiv \Delta_p\mathcal{E}'(\mathrm{t}) - \Delta_p\mathcal{E}'_p + \frac{\grave{\mathrm{i}}^2_p(\mathrm{t}) - \Delta_p^2}{2}\mathcal{E}''_p - \frac{\acute{\mathrm{i}}^2_p(\mathrm{t})}{2}\mathcal{E}''_{p+1} + (\frac{\acute{\mathrm{i}}^3_p(\mathrm{t})}{6} - \frac{\Delta_p\acute{\mathrm{i}}^2_p(\mathrm{t})}{4})\Delta_p\mathsf{y}^{(4)}(\xi_{\mathrm{B}}(t_p))$$

From: $\mathcal{E}_p = 0$; (16), constancy of $\boldsymbol{\omega}\langle\acute{\mathrm{i}}_p(\mathrm{t})/6 - \Delta_p/4\rangle$ if $\mathrm{t} \in \mathfrak{I}_p$; $\int_{\mathsf{x}_p}^{t_p}\mathsf{B}_p(\mathrm{t})\,\mathrm{dt} = 0$ due to $\mathsf{B}_p(t_p) = 0$ affirmed by (18) and to $\mathfrak{R}_{t_p} \equiv (\mathsf{x}_p, \mathsf{x}_{p+1})$; follows

$$\int_{\mathsf{x}_p}^{t_p} \mathsf{B}_p(\mathrm{t})\,\mathrm{dt} = \Delta_p\mathcal{E}(t_p) - \Delta_p\acute{\mathrm{i}}_p(t_p)\mathcal{E}'_p + \frac{\grave{\mathrm{i}}^3_p(t_p) + \Delta_p^3 - 3\Delta_p^2\acute{\mathrm{i}}_p(t_p)}{6}\mathcal{E}''_p -$$
$$\frac{\acute{\mathrm{i}}^3_p(t_p)}{6}\mathcal{E}''_{p+1} + \Delta_p\int_{\mathsf{x}_p}^{t_p}\big(\acute{\mathrm{i}}_p(\mathrm{t})/6 - \Delta_p/4\big)\,\acute{\mathrm{i}}^2_p(\mathrm{t})\,\mathsf{y}^{(4)}(\xi_{\mathrm{B}}(\mathrm{t}))\,\mathrm{dt} = \mathsf{C}_p(t_p) = 0 \quad (19)$$

of which, being $\xi_{\mathrm{C}}(t_p) \in \mathfrak{R}_{t_p}$, is had

$$\mathsf{C}_p(\mathrm{t}) \equiv \Delta_p\mathcal{E}(\mathrm{t}) - \Delta_p\acute{\mathrm{i}}_p(\mathrm{t})\mathcal{E}'_p + \frac{\acute{\mathrm{i}}^3_p(\mathrm{t}) - 3\Delta_p\acute{\mathrm{i}}^2_p(\mathrm{t})}{6}\mathcal{E}''_p - \frac{\acute{\mathrm{i}}^3_p(\mathrm{t})}{6}\mathcal{E}''_{p+1} +$$
$$\Delta_p(\frac{\acute{\mathrm{i}}^4_p(\mathrm{t})}{24} - \frac{\Delta_p\acute{\mathrm{i}}^3_p(\mathrm{t})}{12})\mathsf{y}^{(4)}(\xi_{\mathrm{C}}(\mathrm{t}))$$

By implementing for $\{\mathsf{x}_{p+1}, t_p\}$ a process similar to that for $\{\mathsf{x}_p, t_p\}$ has led to (18) and (19) starting from $\mathsf{A}_p(t_p) = 0$, are obtained $\mathsf{D}_p(t_p) = 0$ and $\mathsf{E}_p(t_p) = 0$ of which

$$\mathsf{D}_p(\mathrm{t}) \equiv \Delta_p \mathcal{E}'(\mathrm{t}) - \Delta_p \mathcal{E}'_{p+1} + \frac{\grave{\mathrm{I}}_p^2(\mathrm{t})}{2}\mathcal{E}''_p + \frac{\Delta_p^2 - \acute{\mathrm{I}}_p^2(\mathrm{t})}{2}\mathcal{E}''_{p+1} + \left(\frac{\acute{\mathrm{I}}_p^3(\mathrm{t})}{6} + \frac{\Delta_p^3}{12} - \frac{\acute{\mathrm{I}}_p^2(\mathrm{t})\Delta_p}{4}\right)\Delta_p \mathsf{y}^{(4)}(\xi_{\mathrm{D}}(\mathrm{t}))$$

$$\mathsf{E}_p(\mathrm{t}) \equiv \Delta_p \mathcal{E}(\mathrm{t}) - \grave{\mathrm{I}}_p(\mathrm{t})\Delta_p \mathcal{E}'_{p+1} + \frac{\grave{\mathrm{I}}_p^3(\mathrm{t})}{6}\mathcal{E}''_p + \frac{3\grave{\mathrm{I}}_p(\mathrm{t})\Delta_p^2 + \Delta_p^3 - \acute{\mathrm{I}}_p^3(\mathrm{t})}{6}\mathcal{E}''_{p+1} + \left(\frac{\acute{\mathrm{I}}_p^4(\mathrm{t}) - \Delta_p^4}{24} + \frac{\Delta_p^2 \grave{\mathrm{I}}_p(\mathrm{t}) - \acute{\mathrm{I}}_p^3(\mathrm{t}) + \Delta_p^3}{12}\Delta_p\right)\Delta_p \mathsf{y}^{(4)}(\xi_{\mathrm{E}}(t_p))$$

where $\xi_{\mathrm{D}}(t_p) \in \mathfrak{R}_{t_p}$ and $\xi_{\mathrm{E}}(t_p) \in \mathfrak{R}_{t_p}$.

The limit of $\mathsf{B}_p(t_p) = 0$ as $t_p \to \mathsf{x}_{p+1}$ gives rise, intending $\zeta_{\mathrm{A}} \in \mathfrak{R}_{t_p}$, to

$$\mathcal{E}'_p - \mathcal{E}'_{p+1} + \frac{\mathcal{E}''_p + \mathcal{E}''_{p+1}}{2}\Delta_p + \frac{\Delta_p^3}{12}\mathsf{y}^{(4)}(\zeta_{\mathrm{A}}) = 0 \tag{20}$$

The limit of $\mathsf{C}_p(t_p) = 0$ as $t_p \to \mathsf{x}_{p+1}$ gives rise, considering also $\mathcal{E}_{p+1} = 0$ and intending $\zeta_{\mathrm{B}} \in \mathfrak{R}_{t_p}$, to

$$\mathcal{E}'_p + \frac{\Delta_p}{3}\mathcal{E}''_p + \frac{\Delta_p}{6}\mathcal{E}''_{p+1} + \frac{\Delta_p^3}{24}\mathsf{y}^{(4)}(\zeta_{\mathrm{B}}) = 0 \tag{21}$$

The limit of $\mathsf{E}_p(t_p) = 0$ as $t_p \to \mathsf{x}_p$ gives rise, intending $\zeta_{\mathrm{C}} \in \mathfrak{R}_{t_p}$, to

$$\mathcal{E}'_{p+1} + \frac{\Delta_p}{6}\mathcal{E}''_p - \frac{\Delta_p}{3}\mathcal{E}''_{p+1} - \frac{\Delta_p^3}{24}\mathsf{y}^{(4)}(\zeta_{\mathrm{C}}) = 0 \tag{22}$$

The Rolle's theorem ([10],[11]) asserts

$$\{f(\mathrm{x}) \in C^0[a,b], f(\mathrm{x}) \in C^1(a,b), f(a) = f(b)\} \Rightarrow \exists \{f'(x) = 0 \parallel x \in (a,b)\} \tag{23}$$

Is placed

$$\mathrm{F}(\mathsf{x}) \equiv \mathcal{E}''(\mathsf{x}) - \frac{\acute{\mathrm{I}}_p(\mathsf{x})\mathcal{E}''_{p+1} - \grave{\mathrm{I}}_p(\mathsf{x})\mathcal{E}''_p}{\Delta_p}$$

This, $\tilde{\mathrm{S}}(\mathsf{x}) \in C^2(\mathfrak{R}_{\mathsf{x}})$ and $\tilde{\mathrm{S}}(\mathsf{x}) \in C^3(\mathfrak{R}_{\mathsf{x}} - \{\mathsf{x}_p; p = 2, \hat{\mathrm{p}} - 1\})$ (due to (17)) entail

$$\mathrm{F}(\mathsf{x}_p) = \mathrm{F}(\mathsf{x}_{p+1}) = 0 \qquad \mathrm{F}(\mathsf{x}) \in C^0(\mathfrak{I}_p) \qquad \mathrm{F}(\mathsf{x}) \in C^1(\mathsf{x}_p, \mathsf{x}_{p+1})$$

This deduction and $\text{Æ}\langle \mathrm{F}(\mathsf{x}), \mathsf{x}_p, \mathsf{x}_{p+1} /\!\!/ f(\mathrm{x}), a, b /\!\!/ (23)\rangle$ entail, intending $\zeta_p \in (\mathsf{x}_p, \mathsf{x}_{p+1})$,

$$\mathcal{E}''_p - \mathcal{E}''_{p+1} + \Delta_p \mathcal{E}^{(3)}(\zeta_p) = 0 \tag{24}$$

Are placed the

$$\{p_{\mathsf{A}}, p_{\mathsf{B}}\} \subseteq \{p = 1, \hat{\mathrm{p}} - 1\} \qquad p_{\mathsf{B}} > p_{\mathsf{A}} \qquad \underline{\mathsf{J}} \equiv \{\mathsf{J}_p; p = 1, \hat{\mathrm{p}} - 1\} \qquad \underline{\mathsf{L}} \equiv \{\mathsf{L}_p; p = 1, \hat{\mathrm{p}} - 1\}$$
$$\{\mathsf{J}_p = 0; \forall p \notin \{\mathrm{p} = p_{\mathsf{A}}, p_{\mathsf{B}}\}\} \qquad \{\mathsf{J}_p = \{1 \dot{\vee} 2\}; \forall p \in \{\mathrm{p} = p_{\mathsf{A}}, p_{\mathsf{B}}\}\}$$
$$\{\mathsf{u}, \mathsf{v}\} \subseteq \{\mathrm{p} = p_{\mathsf{A}}, p_{\mathsf{B}}\} \qquad \{\mathsf{L}_p = 0; \forall \{p \neq \mathsf{u}\} \bigvee \{p \neq \mathsf{v}\}\} \qquad \mathsf{L}_{\mathsf{u}} = \mathsf{L}_{\mathsf{v}} = 1 \tag{25}$$

The writing of $\mathsf{A}_p(t_p) = 0$, $\mathsf{B}_p(t_p) = 0$, $\mathsf{C}_p(t_p) = 0$, $\mathsf{D}_p(t_p) = 0$, $\mathsf{E}_p(t_p) = 0$, (20), (21), (22) and (24) for each of $\{p = 1, \hat{\mathrm{p}}-1\}$, subordinated to (25) in the sense that these specify as follows those that are replaced by the same number of expressions all equal to $0 = 0$, gives rise to the homogeneous linear system $\underline{\underline{\mathcal{M}}} \cdot \underline{\mathcal{V}} = \underline{0}_{\hat{\mathsf{m}}}$ (of which $\underline{0}\langle \mathrm{A} \rangle \equiv [0; i = 1, \mathrm{A}]$) defined by $\underline{\underline{\mathcal{M}}} \equiv [\mathcal{M}_{rc}; r = 1, \hat{\mathsf{m}}; c = 1, \hat{\mathsf{n}}]$, $\hat{\mathsf{m}} \equiv 9(\hat{\mathrm{p}} - 1)$ and $\hat{\mathsf{n}} \equiv 14\hat{\mathrm{p}} - 12$, by

$$\underline{\mathcal{V}} \equiv [\mathcal{V}_c; c = 1, \hat{\mathsf{n}}] = \Big[\{\mathcal{E}(t_p); p = 1, \hat{\mathrm{p}} - 1\}, \{\mathcal{E}'(t_p); p = 1, \hat{\mathrm{p}} - 1\}, \{\mathcal{E}''(t_p); p = 1, \hat{\mathrm{p}} - 1\},$$
$$\{\mathcal{E}'_p; p = 1, \hat{\mathrm{p}}\}, \{\mathcal{E}''_p; p = 1, \hat{\mathrm{p}}\}, \{\mathcal{E}^{(3)}(\zeta_p); p = 1, \hat{\mathrm{p}} - 1\}, \underline{\mathsf{Q}}\Big]$$

of which $\zeta_p \in (\mathsf{x}_p, \mathsf{x}_{p+1})$ and $\underline{\mathsf{Q}} \equiv \{\mathsf{Q}_{qp}; p = 1, \hat{\mathrm{p}} - 1; q = 1, 8\}$ with Q_{qp} a q-th value of $\mathsf{y}^{(4)}(\mathsf{x})$ in $(\mathsf{x}_p, \mathsf{x}_{p+1})$, and by the nullity of all the elements of $\underline{\underline{\mathcal{M}}}$ that are not defined by

$$\mathcal{M}\Big\langle \mathsf{H}\langle p, 1\rangle, \mathsf{K}\langle 2, 0\rangle + p\Big\rangle = \Delta_p \delta_{1\mathsf{J}_p} \qquad \mathcal{M}\Big\langle \mathsf{H}\langle p, 1\rangle, \mathsf{K}\langle 3, 1\rangle + p\Big\rangle = \grave{\imath}_p(t_p)\delta_{1\mathsf{J}_p}$$
$$\mathcal{M}\Big\langle \mathsf{H}\langle p, 1\rangle, \mathsf{K}\langle 3, 1\rangle + p + 1\Big\rangle = -\acute{\imath}_p(t_p)\delta_{1\mathsf{J}_p}$$
$$\mathcal{M}\Big\langle \mathsf{H}\langle p, 1\rangle, \mathsf{K}\langle 4, 2\rangle + p\Big\rangle = -\Delta_p \acute{\imath}_p(t_p)\grave{\imath}_p(t_p)\delta_{1\mathsf{J}_p}/2$$

$$\mathcal{M}\Big\langle \mathsf{H}\langle p, 2\rangle, \mathsf{K}\langle 1, 0\rangle + p\Big\rangle = \Delta_p \delta_{1\mathsf{J}_p} \qquad \mathcal{M}\Big\langle \mathsf{H}\langle p, 2\rangle, \mathsf{K}\langle 3, 0\rangle + p\Big\rangle = -\Delta_p \delta_{1\mathsf{J}_p}$$
$$\mathcal{M}\Big\langle \mathsf{H}\langle p, 2\rangle, \mathsf{K}\langle 3, 1\rangle + p\Big\rangle = \big(\grave{\imath}^2_p(t_p) - \Delta^2_p\big)\delta_{1\mathsf{J}_p}/2$$
$$\mathcal{M}\Big\langle \mathsf{H}\langle p, 2\rangle, \mathsf{K}\langle 3, 1\rangle + p + 1\Big\rangle = -\acute{\imath}^2_p(t_p)\delta_{1\mathsf{J}_p}/2$$
$$\mathcal{M}\Big\langle \mathsf{H}\langle p, 2\rangle, \mathsf{K}\langle 5, 2\rangle + p\Big\rangle = \Delta_p \acute{\imath}^2_p(t_p)\big(\Delta_p/4 - \acute{\imath}_p(t_p)/6\big)\delta_{1\mathsf{J}_p}$$

$$\mathcal{M}\Big\langle \mathsf{H}\langle p, 3\rangle, p\Big\rangle = \Delta_p \delta_{1\mathsf{J}_p} \qquad \mathcal{M}\Big\langle \mathsf{H}\langle p, 3\rangle, \mathsf{K}\langle 3, 0\rangle + p\Big\rangle = -\Delta_p \delta_{1\mathsf{J}_p}\acute{\imath}_p(t_p)$$
$$\mathcal{M}\Big\langle \mathsf{H}\langle p, 3\rangle, \mathsf{K}\langle 3, 1\rangle + p\Big\rangle = \big(\acute{\imath}^3_p(t_p) - 3\Delta_p \acute{\imath}^2_p(t_p)\big)\delta_{1\mathsf{J}_p}/6$$
$$\mathcal{M}\Big\langle \mathsf{H}\langle p, 3\rangle, \mathsf{K}\langle 3, 1\rangle + p + 1\Big\rangle = -\acute{\imath}^3_p(t_p)\delta_{1\mathsf{J}_p}/6$$
$$\mathcal{M}\Big\langle \mathsf{H}\langle p, 3\rangle, \mathsf{K}\langle 6, 2\rangle + p\Big\rangle = \Delta_p \acute{\imath}^3_p(t_p)\big(\Delta_p/12 - \acute{\imath}_p(t_p)/24\big)\delta_{1\mathsf{J}_p}$$

$$\mathcal{M}\Big\langle \mathsf{H}\langle p, 4\rangle, \mathsf{K}\langle 1, 0\rangle + p\Big\rangle = \Delta_p \delta_{1\mathsf{J}_p} \qquad \mathcal{M}\Big\langle \mathsf{H}\langle p, 4\rangle, \mathsf{K}\langle 3, 0\rangle + p + 1\Big\rangle = -\Delta_p \delta_{1\mathsf{J}_p}$$
$$\mathcal{M}\Big\langle \mathsf{H}\langle p, 4\rangle, \mathsf{K}\langle 3, 1\rangle + p\Big\rangle = \grave{\imath}^2_p(t_p)\delta_{1\mathsf{J}_p}/2$$
$$\mathcal{M}\Big\langle \mathsf{H}\langle p, 4\rangle, \mathsf{K}\langle 3, 1\rangle + p + 1\Big\rangle = \big(\Delta^2_p - \acute{\imath}^2_p(t_p)\big)\delta_{1\mathsf{J}_p}/2$$
$$\mathcal{M}\Big\langle \mathsf{H}\langle p, 4\rangle, \mathsf{K}\langle 7, 2\rangle + p\Big\rangle = \Delta_p\big(\Delta_p \acute{\imath}^2_p(t_p)/4 - \Delta^3_p/12 - \acute{\imath}^3_p(t_p)/6\big)\delta_{1\mathsf{J}_p}$$

$$\mathcal{M}\Big\langle \mathsf{H}\langle p,5\rangle, p\Big\rangle = \Delta_p \delta_{1\mathsf{J}_p} \qquad \mathcal{M}\Big\langle \mathsf{H}\langle p,5\rangle, \mathsf{K}\langle 3,0\rangle + p + 1\Big\rangle = -\Delta_p \grave{\imath}_p(t)\delta_{1\mathsf{J}_p}$$
$$\mathcal{M}\Big\langle \mathsf{H}\langle p,5\rangle, \mathsf{K}\langle 3,1\rangle + p\Big\rangle = \grave{\imath}_p^3(t)\delta_{1\mathsf{J}_p}/6$$
$$\mathcal{M}\Big\langle \mathsf{H}\langle p,5\rangle, \mathsf{K}\langle 3,1\rangle + p + 1\Big\rangle = \big(3\Delta_p^2 \grave{\imath}_p(t) + \Delta_p^3 - \acute{\imath}_p^3(t)\big)\delta_{1\mathsf{J}_p}/6$$
$$\mathcal{M}\Big\langle \mathsf{H}\langle p,5\rangle, \mathsf{K}\langle 8,2\rangle + p\Big\rangle = \Delta_p\Big(\big(\Delta_p^4 - \acute{\imath}_p^4(t)\big)/24 - \Delta_p\big(\Delta_p^2 \grave{\imath}_p(t) - \acute{\imath}_p^3(t) + \Delta_p^3\big)/12\Big)\delta_{1\mathsf{J}_p}$$

$$\mathcal{M}\Big\langle \mathsf{H}\langle p,6\rangle, \mathsf{K}\langle 3,0\rangle + p\Big\rangle = \delta_{2\mathsf{J}_p} \qquad \mathcal{M}\Big\langle \mathsf{H}\langle p,6\rangle, \mathsf{K}\langle 3,0\rangle + p + 1\Big\rangle = -\delta_{2\mathsf{J}_p}$$
$$\mathcal{M}\Big\langle \mathsf{H}\langle p,6\rangle, \mathsf{K}\langle 3,1\rangle + p\Big\rangle = \Delta_p\delta_{2\mathsf{J}_p}/2 \qquad \mathcal{M}\Big\langle \mathsf{H}\langle p,6\rangle, \mathsf{K}\langle 3,1\rangle + p + 1\Big\rangle = \Delta_p\delta_{2\mathsf{J}_p}/2$$
$$\mathcal{M}\Big\langle \mathsf{H}\langle p,6\rangle, \mathsf{K}\langle 9,2\rangle + p\Big\rangle = -\Delta_p^3\delta_{2\mathsf{J}_p}/12$$

$$\mathcal{M}\Big\langle \mathsf{H}\langle p,7\rangle, \mathsf{K}\langle 3,0\rangle + p\Big\rangle = \delta_{2\mathsf{J}_p} \qquad \mathcal{M}\Big\langle \mathsf{H}\langle p,7\rangle, \mathsf{K}\langle 3,1\rangle + p\Big\rangle = \Delta_p\delta_{2\mathsf{J}_p}/3$$
$$\mathcal{M}\Big\langle \mathsf{H}\langle p,7\rangle, \mathsf{K}\langle 3,1\rangle + p + 1\Big\rangle = \Delta_p\delta_{2\mathsf{J}_p}/6 \qquad \mathcal{M}\Big\langle \mathsf{H}\langle p,7\rangle, \mathsf{K}\langle 10,2\rangle + p\Big\rangle = -\Delta_p^3\delta_{2\mathsf{J}_p}/24$$

$$\mathcal{M}\Big\langle \mathsf{H}\langle p,8\rangle, \mathsf{K}\langle 3,0\rangle + p + 1\Big\rangle = \delta_{2\mathsf{J}_p} \qquad \mathcal{M}\Big\langle \mathsf{H}\langle p,8\rangle, \mathsf{K}\langle 3,1\rangle + p\Big\rangle = -\Delta_p\delta_{2\mathsf{J}_p}/6$$
$$\mathcal{M}\Big\langle \mathsf{H}\langle p,8\rangle, \mathsf{K}\langle 3,1\rangle + p + 1\Big\rangle = -\Delta_p\delta_{2\mathsf{J}_p}/3 \qquad \mathcal{M}\Big\langle \mathsf{H}\langle p,8\rangle, \mathsf{K}\langle 11,2\rangle + p\Big\rangle = \Delta_p^3\delta_{2\mathsf{J}_p}/24$$

$$\mathcal{M}\Big\langle \mathsf{H}\langle p,9\rangle, \mathsf{K}\langle 3,1\rangle + p\Big\rangle = \delta_{1\mathsf{L}_p} \qquad \mathcal{M}\Big\langle \mathsf{H}\langle p,9\rangle, \mathsf{K}\langle 3,1\rangle + p + 1\Big\rangle = -\delta_{1\mathsf{L}_p}$$
$$\mathcal{M}\Big\langle \mathsf{H}\langle p,9\rangle, \mathsf{K}\langle 3,2\rangle + p\Big\rangle = -\Delta_p\delta_{1\mathsf{L}_p}$$

of which $p \in \{p = 1, \hat{\mathrm{p}} - 1\}$, $\mathsf{H}\langle a,b\rangle \equiv 9(a-1) + b$ and $\mathsf{K}\langle a,b\rangle \equiv a(\hat{\mathrm{p}} - 1) + b\hat{\mathrm{p}}$.

From $\underline{\underline{\mathcal{M}}}$ is obtained a matrix $\underline{\underline{\mathsf{N}}}$ with the following steps:

1. are posed the $\underline{\underline{\mathsf{N}}} \equiv [\mathsf{N}_{rc}; r = 1, \hat{\mathsf{m}}; c = 1, \hat{\mathsf{n}}] \equiv \underline{\underline{\mathcal{M}}}$, $\{r_m; m = 1, \hat{\mathsf{m}}\} = \{m = 1, \hat{\mathsf{m}}\}$, $\{c_n; n = 1, \hat{\mathsf{n}}\} = \{n = 1, \hat{\mathsf{n}}\}$, $\mathrm{R}{=}1$;
2. if $\mathrm{R} = \mathsf{K}\langle 3,2\rangle + 1$ is executed step 8;
3. is posed $\mathrm{P} = \max\{|\mathsf{N}\langle r_m, c_n\rangle|; m = \mathrm{R}, \hat{\mathsf{m}}; n = \mathrm{R}, \mathsf{K}\langle 3,2\rangle\}$;
4. if $\mathrm{P} = 0$ is executed step 8;
5. is posed $\{\mathrm{r}, \mathrm{c}\} = \{m, n \parallel \mathrm{P} = |\mathsf{N}\langle r_m, c_n\rangle|\}$, are exchanged the values between r_{R} and r_{r} and between c_{R} and c_{c};
6. are replaced the $\{\mathsf{N}\langle r_m, c_n\rangle; m = \mathrm{R} + 1, \hat{\mathsf{m}}; n = \mathrm{R} + 1, \hat{\mathsf{n}}\}$ with the respective $\{\mathsf{N}\langle r_m, c_n\rangle - \mathsf{N}\langle r_m, c_{\mathrm{R}}\rangle\mathsf{N}\langle r_{\mathrm{R}}, c_n\rangle/\mathsf{N}\langle r_{\mathrm{R}}, c_{\mathrm{R}}\rangle; m = \mathrm{R} + 1, \hat{\mathsf{m}}; n = \mathrm{R}, \hat{\mathsf{n}}\}$;
7. R is incremented by 1 and it is returned to step 2;
8. R is decreased by 1 and are replaced the $\{\mathsf{N}\langle r_m, c_n\rangle; n = m, \hat{\mathsf{n}}; m = 1, \mathrm{R}\}$ with the respective $\{\mathsf{N}\langle r_m, c_n\rangle/\mathsf{N}\langle r_m, c_n\rangle; n = m, \hat{\mathsf{n}}; m = 1, \mathrm{R}\}$;

9. are performed the iterations indicated by $\{k = 2, \mathrm{R}\}$ and at the k-th are replaced the $\{\mathsf{N}\langle r_m, c_n\rangle; m = 1, k-1; n = k, \hat{\mathsf{n}}\}$ with the respective $\{\mathsf{N}\langle r_m, c_n\rangle - \mathsf{N}\langle r_k, c_n\rangle \mathsf{N}\langle r_m, c_k\rangle; m = 1, k-1; n = k, \hat{\mathsf{n}}\}$.

The $\underline{\underline{\mathsf{N}}}$ that is had after this execution verifies still the $\underline{\underline{\mathsf{N}}} \cdot \underline{\mathcal{V}} = \underline{0}_{\hat{\mathsf{m}}}$ as when it was placed the $\underline{\underline{\mathsf{N}}} \equiv \underline{\underline{\mathcal{M}}}$, with the advantage that each of the rows, indicated by $\{r_k; k = 1, \mathrm{R}\}$, of this additional homogeneous linear system gives rise to a corresponding

$$\mathcal{V}_{c_k} = -\sum_{q=1}^{8}\sum_{p=1}^{\hat{\mathsf{p}}-1} \mathsf{N}\big\langle r_k, \mathsf{K}\langle q+3, 2\rangle + p\big\rangle \mathsf{Q}_{qp} - \mathsf{N}\big\langle r_k, \mathsf{K}\langle 3, 2\rangle + \mathsf{u}\big\rangle \mathcal{E}^{(3)}(\zeta_{\mathsf{u}}) - \mathsf{N}\big\langle r_k, \mathsf{K}\langle 3, 2\rangle + \mathsf{v}\big\rangle \mathcal{E}^{(3)}(\zeta_{\mathsf{v}}) \quad (26)$$

of which

$$\mathcal{V}_{c_k} \in \Big\{\{\mathcal{E}(t_p); p = p_{\mathtt{A}}, p_{\mathtt{B}}\}, \{\mathcal{E}'(t_p); p = p_{\mathtt{A}}, p_{\mathtt{B}}\}, \{\mathcal{E}''(t_p); p = p_{\mathtt{A}}, p_{\mathtt{B}}\}, \{\mathcal{E}'_p; p = p_{\mathtt{A}}, p_{\mathtt{B}}+1\}, \{\mathcal{E}''_p; p = p_{\mathtt{A}}, p_{\mathtt{B}}+1\}\Big\}$$

$$\Big\{\exists\,\{k \parallel \mathcal{V}_{c_k} \equiv \mathcal{E}'_p\}, \exists\,\{k \parallel \mathcal{V}_{c_k} \equiv \mathcal{E}''_p\}; p = p_{\mathtt{A}}, p_{\mathtt{B}}+1\Big\}$$

This, (24) and $\mathsf{p} \in \{p = p_{\mathtt{A}}, p_{\mathtt{B}}+1\}$ entail

$$\mathcal{E}''_{\mathsf{p}} = \alpha_{\mathsf{p}} + \beta_{\mathsf{up}}(\mathcal{E}''_{\mathsf{u}+1} - \mathcal{E}''_{\mathsf{u}}) + \beta_{\mathsf{vp}}(\mathcal{E}''_{\mathsf{v}+1} - \mathcal{E}''_{\mathsf{v}})$$

of which

$$\alpha_{\mathsf{p}} \equiv -\sum_{q=1}^{8}\sum_{p=1}^{\hat{\mathsf{p}}-1} \mathsf{N}\big\langle r_{\mathsf{a_p}}, \mathsf{K}\langle q+3, 2\rangle + p\big\rangle \mathsf{Q}_{qp} \qquad \beta_{\mathsf{up}} \equiv -\frac{\mathsf{N}\big\langle r_{\mathsf{a_p}}, \mathsf{K}\langle 3, 2\rangle + \mathsf{u}\big\rangle}{\Delta_{\mathsf{u}}}$$

where $\mathsf{a_p} \equiv \{k \parallel c_k = \mathsf{K}\langle 3, 1\rangle + \mathsf{p}\}$.

This implies

$$\mathcal{E}''_{\mathsf{p}+1} - \mathcal{E}''_{\mathsf{p}} = \tilde{\alpha}_{\mathsf{p}} + \grave{\beta}_{\mathsf{up}}(\mathcal{E}''_{\mathsf{u}+1} - \mathcal{E}''_{\mathsf{u}}) + \grave{\beta}_{\mathsf{vp}}(\mathcal{E}''_{\mathsf{v}+1} - \mathcal{E}''_{\mathsf{v}})$$

of which $\tilde{\alpha}_{\mathsf{p}} \equiv \alpha_{\mathsf{p}+1} - \alpha_{\mathsf{p}}$, $\grave{\beta}_{\mathsf{up}} \equiv \beta_{\mathsf{u,p}+1} - \beta_{\mathsf{up}}$, and that, by introducing $\mathsf{p} \equiv \mathsf{u}$ or $\mathsf{p} \equiv \mathsf{v}$, gives rise, intending $\acute{\beta}_{\mathsf{uu}} \equiv 1 - \grave{\beta}_{\mathsf{uu}}$, to the respective

$$\acute{\beta}_{\mathsf{uu}}(\mathcal{E}''_{\mathsf{u}+1} - \mathcal{E}''_{\mathsf{u}}) - \grave{\beta}_{\mathsf{vu}}(\mathcal{E}''_{\mathsf{v}+1} - \mathcal{E}''_{\mathsf{v}}) = \tilde{\alpha}_{\mathsf{u}} \qquad \grave{\beta}_{\mathsf{uv}}(\mathcal{E}''_{\mathsf{u}+1} - \mathcal{E}''_{\mathsf{u}}) - \acute{\beta}_{\mathsf{vv}}(\mathcal{E}''_{\mathsf{v}+1} - \mathcal{E}''_{\mathsf{v}}) = -\tilde{\alpha}_{\mathsf{v}}$$

By solving with the Cramer's method ([10]) the system in the unknowns $\mathcal{E}''_{\mathsf{u}+1} - \mathcal{E}''_{\mathsf{u}}$ and $\mathcal{E}''_{\mathsf{v}+1} - \mathcal{E}''_{\mathsf{v}}$ constituted by these two equations, is had

$$\mathcal{E}''_{\mathsf{u}+1} - \mathcal{E}''_{\mathsf{u}} = \frac{\grave{\beta}_{\mathsf{vu}}\tilde{\alpha}_{\mathsf{v}} + \acute{\beta}_{\mathsf{vv}}\tilde{\alpha}_{\mathsf{u}}}{\tilde{\theta}_{\mathsf{uv}}} \qquad \mathcal{E}''_{\mathsf{v}+1} - \mathcal{E}''_{\mathsf{v}} = \frac{\grave{\beta}_{\mathsf{uv}}\tilde{\alpha}_{\mathsf{u}} + \acute{\beta}_{\mathsf{uu}}\tilde{\alpha}_{\mathsf{v}}}{\tilde{\theta}_{\mathsf{uv}}}$$

of which $\tilde{\theta}_{uv} \equiv \acute{\beta}_{vv}\acute{\beta}_{uu} - \grave{\beta}_{vu}\grave{\beta}_{uv}$.

This and the fact that occur $\grave{\beta}_{uu} = \grave{\beta}_{vv} = -1$ and $\grave{\beta}_{uv} = \grave{\beta}_{vu} = 0$, imply $\mathcal{E}''_{u+1} - \mathcal{E}''_{u} = \tilde{\alpha}_u/2$ and $\mathcal{E}''_{v+1} - \mathcal{E}''_{v} = \tilde{\alpha}_v/2$. These and (24) entail $\mathcal{E}^{(3)}(\zeta_u) = \tilde{\alpha}_u/(2\Delta_u)$ and $\mathcal{E}^{(3)}(\zeta_v) = \tilde{\alpha}_v/(2\Delta_v)$. Introducing these and $\tilde{\alpha}_p = \tilde{\alpha}_p(\underline{Q})$ in (26), is had $\mathcal{V}_{c_k} = -\sum_{q=1}^{8}\sum_{p=1}^{\hat{p}-1}\tilde{N}_{kuvqp}Q_{qp}$ of which

$$\tilde{N}_{kuvqp} \equiv N\langle r_k, K\langle q+3,2\rangle + p\rangle + \frac{N\langle r_{a_u}, K\langle q+3,2\rangle + p\rangle - N\langle r_{a_{u+1}}, K\langle q+3,2\rangle + p\rangle}{2\Delta_u} N\langle r_k, K\langle 3,2\rangle + u\rangle + \frac{N\langle r_{a_v}, K\langle q+3,2\rangle + p\rangle - N\langle r_{a_{v+1}}, K\langle q+3,2\rangle + p\rangle}{2\Delta_v} N\langle r_k, K\langle 3,2\rangle + v\rangle$$

and that gives rise to

$$|\mathcal{V}_{c_k}| = \left|\sum_{q=1}^{8}\sum_{p=1}^{\hat{p}-1}\tilde{N}_{kuvqp}Q_{qp}\right| \leq \sum_{q=1}^{8}\sum_{p=1}^{\hat{p}-1}\left|\tilde{N}_{kuvqp}Q_{qp}\right| \leq \sum_{p=1}^{\hat{p}-1}\Phi_p\langle y^{(4)}\rangle\sum_{q=1}^{8}\left|\tilde{N}_{kuvqp}\right| \leq \Phi\langle y^{(4)}\rangle\sum_{p=1}^{\hat{p}-1}\sum_{q=1}^{8}\left|\tilde{N}_{kuvqp}\right| \quad (27)$$

of which

$$\Phi_p\langle y^{(4)}\rangle \equiv \max\{|y^{(4)}(x)| \,/\!\!/\, x \in \mathfrak{R}_{t_p}\} \qquad \Phi\langle y^{(4)}\rangle \equiv \max\{|y^{(4)}(x)| \,/\!\!/\, x \in (x_1, x_{\hat{p}})\}$$

Each different choice of the p_A, p_B, $\underline{J}$, u, v and $\{t_p; p = 1, \hat{p}-1\}$ gives rise to a correspondingly different (27) which can possibly enable a further lower upper bound for some of the $\{|\mathcal{V}_c|; c = 1, K\langle 3,2\rangle\}$. Hence for $\{|\mathcal{E}'_p|; p = 1, \hat{p}\}$ is in particular possible to proceed as follows. Is placed $\{\mathrm{K}_p \equiv \infty; p = 1, \hat{p}\}$ and are chosen two natural numbers $\mathrm{N_{IT}}$ and $\hat{\mathrm{N}}_{\mathrm{IN}}$ of which $\hat{\mathrm{N}}_{\mathrm{IN}} \geq 2$. Are carried the iterations indicated by $\{\mathrm{N_{IN}} = 2, \hat{\mathrm{N}}_{\mathrm{IN}}\}$. For every $\mathrm{N_{IN}}$, if $\hat{p} - \mathrm{N_{IN}} \geq 1$, are carried the iterations indicated by $\{p_A = 1, \hat{p} - \mathrm{N_{IN}}\}$. In the p_A-th of these iterations, is placed $p_B = p_A + \mathrm{N_{IN}} - 1$ and are carried the iterations indicated by $\{\mathrm{I_t} = 1, N_{\mathrm{IT}}\}$ of which $\{N_{\mathrm{IT}} > \mathrm{N_{IT}}; \forall\{p_A = 1\} \veebar \{p_A = \hat{p} - \mathrm{N_{IN}}\}\}$ and $\{N_{\mathrm{IT}} = \mathrm{N_{IT}}; \forall p_A \in \{p = 2, \hat{p} - \mathrm{N_{IN}} - 1\}\}$. For every $\mathrm{I_t}$, are chosen $\{J_p, t_p; p = p_A, p_B\}$ and $\{u, v\}$ in conformity with (25), are calculated the corresponding $\{K_p; p = p_A, p_B + 1\}$ using $K_p = \sum_{p=1}^{\hat{p}-1}\sum_{q=1}^{8}|\tilde{N}_{kuvqp}|$ and $k \equiv \{k \parallel c_k = K\langle 3,2\rangle + p\}$, and for every $\{p = p_A, p_B + 1\}$ is replaced K_p with K_p if $K_p < \mathrm{K}_p$. After these steps is had

$$\{|\mathcal{E}'_p| \leq \mathrm{K}_p\Phi\langle y^{(4)}\rangle; p = 1, \hat{p}\} \quad (28)$$

which generally improves with the increase of $\hat{\mathrm{N}}_{\mathrm{IN}}$ and, for the evident greater influence on $\mathcal{E}'_p$ of those of the $\{\mathfrak{I}_p; p = 1, \hat{p}-1\}$ that are closer to it, even more with the increase of $\mathrm{N_{IT}}$.

3 The maximum absolute value of a derivative defined on a curve of the multidimensional Euclidean space

A vector, also called free vector, is a straight line segment, defined by a direction (it can lie on whichever straight line identifiable as an element of a corresponding infinite set of parallel straight lines), by a sense (its extreme points are distinct as initial and terminal) and by a length (or magnitude) which is its measure. A vector applied at a point is a vector which has such point as initial. A versor, also called unit vector, is a vector which has unitary magnitude.

For a vector $\vec{x}$, $|\vec{x}|$ (of which $|\vec{x}| \equiv x \geq 0$) and $\vec{\upsilon}\langle x \rangle$ indicate respectively magnitude and versor of $\vec{x}$, being $\vec{\upsilon}_x$ the vector which has same direction and sense of $\vec{x}$ but unitary magnitude. The product $a\vec{x}$, of the real number a and $\vec{x}$, is a vector that has same direction of $\vec{x}$, same sense of $\vec{x}$ if $a > 0$ or opposite to that of $\vec{x}$ if $a < 0$, and magnitude of which $|a\vec{x}| = |a||\vec{x}|$; and from this follows $\vec{x} = x\vec{\upsilon}_x$. The scalar product $\vec{A} \cdot \vec{B}$ of the vectors $\vec{A}$ and $\vec{B}$ is defined by $\vec{A} \cdot \vec{B} \equiv AB\cos\alpha_{AB}$ where α_{AB} is the smaller angle between $\vec{A}$ and $\vec{B}$ when these are specified as vectors applied at a same point.

Are placed $\mathbb{R}^{\hat{n}} \equiv \prod_{n=1}^{\hat{n}} \mathbb{R}^1$, $\mathbb{R}^1 \equiv \mathbb{R} \equiv (-\infty, \infty)$, and thus (referring to section 2.4.1 of [1]) is had $\mathbb{R}^{\hat{n}} \leftrightarrow \mathfrak{S}^{\hat{n}}$ with $\mathfrak{S}^{\hat{n}}$ a $\hat{n}$-dimensional Euclidean space. It is intended that $\mathfrak{S}^{\hat{n}}$ is equipped with a orthogonal Cartesian reference system, which has coordinates $\underline{x}$ each measured on the respective coordinated axes, and which has coordinated versors $\underline{\vec{\upsilon}}$, of which $\underline{\vec{\upsilon}} \equiv \{\vec{\upsilon}_n; n = 1, \hat{n}\}$, each having direction and sense of the respective coordinate axis. A $\vec{x} \equiv \sum_{n=1}^{\hat{n}} x_n \vec{\upsilon}_n$ defines $\vec{x}$ as a vector of $\mathfrak{S}^{\hat{n}}$ which has the components $\underline{x}$. Is had $\vec{\upsilon}_n \cdot \vec{\upsilon}_{\mathrm{n}} = \delta_{n\mathrm{n}}$, $\vec{x} \cdot \vec{\upsilon}_x = x$,

$$\vec{x} \cdot \vec{x} = x^2 = (\sum_{n=1}^{\hat{n}} x_n \vec{\upsilon}_n)^2 = \sum_{n=1}^{\hat{n}} \sum_{\mathrm{n}=1}^{\hat{n}} x_n x_{\mathrm{n}} \vec{\upsilon}_n \vec{\upsilon}_{\mathrm{n}} = \sum_{n=1}^{\hat{n}} x_n^2$$

whose $x^2 = \sum_{n=1}^{\hat{n}} x_n^2$ generalizes the Pythagorean theorem.

Are placed $\{\underline{\mathtt{c}} \leftrightarrow [\check{\mathtt{c}}, \hat{\mathtt{c}}]\} \subseteq \{\mathfrak{S}^1 \leftrightarrow \mathbb{R}^1\}$ and $\underline{\mathtt{c}} \subset \mathfrak{S}^{\hat{n}}$. These define $\underline{\mathtt{c}}$ as a curve lying in $\mathfrak{S}^{\hat{n}}$ and which does not intersect itself. Such $\underline{\mathtt{c}}$ is identified by its parametric functions $\underline{ж}(\mathtt{c})$ of which $\underline{ж}(\mathtt{c}) \equiv \{ж_n(\mathtt{c}); n = 1, \hat{n}\}$ and defined in $\mathfrak{R}\langle \mathtt{c} \rangle$ of which $\mathfrak{R}_{\mathtt{c}} \equiv [\check{\mathtt{c}}, \hat{\mathtt{c}}]$, being $\mathtt{c}$ the curvilinear abscissa measured on $\underline{\mathtt{c}}$ in the sense that $\mathtt{c} - \check{\mathtt{c}}$ is the length of the section of $\underline{\mathtt{c}}$ which has $\underline{ж}(\check{\mathtt{c}})$ and $\underline{ж}(\mathtt{c})$ as extreme points. These $\underline{ж}(\mathtt{c})$ constitute the parametric equations of $\underline{\mathtt{c}}$ i.e. the $\underline{ж} = \underline{ж}(\mathtt{c})$ of which $\underline{ж} \equiv \{ж_n; n = 1, \hat{n}\} \in \underline{\mathtt{c}}$.

The versor $\vec{\tau}$, tangent to $\underline{\mathtt{c}}$ at the point $\underline{ж}$, is expressed by $\vec{\tau} = \vec{\tau}(\mathtt{c}) \equiv \sum_{n=1}^{\hat{n}} \tau_n(\mathtt{c})\vec{\upsilon}_n$ of which $\tau_n = ж_n'$ with τ_n the n-th direction cosine of the tangent to $\underline{\mathtt{c}}$ at $\underline{ж}$ and oriented concordantly with increasing $\mathtt{c}$.

A $\underline{\mathtt{c}}$ is regular if $\{ж_n(\mathtt{c}) \in C^1(\mathfrak{R}_{\mathtt{c}}); n = 1, \hat{n}\}$ and thus, in the graphical aspect, if is, beyond that continuous, also devoid of angular points or cusps where the function $\vec{\tau}(\mathtt{c})$ would have a jump.

The said $\sum_{n=1}^{\hat{n}} x_n^2 = x^2$ has the specification $\sum_{n=1}^{\hat{n}} \tau_n^2 = 1$. Multiplying this by $(\mathrm{d}\mathtt{c})^2$ and considering $\tau_n = \mathrm{d}ж_n / \mathrm{d}\mathtt{c}$ is deduced $(\mathrm{d}\mathtt{c})^2 = \sum_{n=1}^{\hat{n}} (\mathrm{d}ж_n)^2$ (valid for a regular $\underline{\mathtt{c}}$).

Is placed $\mathsf{f}(\mathsf{c}) \equiv f(\underline{ж}(\mathsf{c}))$. From: this; the known rules of derivation of a composite function, $\tau_n = ж'_n$; follows

$$\mathsf{f}'(\mathsf{c}) = \frac{\mathrm{d}f(\underline{ж}(\mathsf{c}))}{\mathrm{d}\mathsf{c}} = \sum_{n=1}^{\hat{n}} \frac{\partial f(\underline{ж})}{\partial ж_n}\tau_n = \nabla f(\underline{ж}) \cdot \vec{\tau} \tag{29}$$

where $\mathsf{f}'(\mathsf{c})$ is the directional derivative of $f(\underline{x})$ at the point $\underline{ж}(\mathsf{c})$ according to direction and sense of $\vec{\tau}$, the vector $\nabla f(\underline{ж})$ is the gradient of $f(\underline{ж})$ defined by $\nabla f(\underline{x}) \equiv \sum_{n=1}^{\hat{n}} \vec{\upsilon}_n \partial f(\underline{x})/\partial x_n$, and of which $\{\partial f(\underline{x})/\partial x_n = \partial f(\underline{ж})/\partial ж_n; \forall \underline{x} \equiv \underline{ж}\}$.

The h-th derivative of the product of two functions $\mathtt{f}(\mathtt{x})$ and $\mathtt{g}(\mathtt{x})$ is expressed by the Leibniz's rule ([10],[11])

$$(\mathtt{fg})^{(h)} = \sum_{k=0}^{h} \binom{h}{k} \mathtt{f}^{(h-k)} \mathtt{g}^{(k)} \tag{30}$$

where $\binom{h}{k}$ is the binomial coefficient defined by

$$\binom{\mathrm{N}}{\mathrm{K}} \equiv \frac{\mathrm{N}!}{\mathrm{K}!(\mathrm{N}-\mathrm{K})!} = \frac{\prod_{i=\mathrm{N}-\mathrm{K}+1}^{\mathrm{N}} i}{\mathrm{K}!}$$

The symbol "…" generally implies other symbols considered evident, and in particular when it is inserted in a sequence indicates that this is constituted by elements that vary from the first to the last neatly with the next trend indicated by the first two, thus having $\{s_1, s_2, \ldots s_k \parallel k = 1\} \equiv s_1$ and $\{s_1, s_2, \ldots s_k \parallel k = 2\} \equiv \{s_1, s_2\}$.

Are placed $o \geq 1$ and

$$\mathsf{f}_{\{n_i; i=1,a\}} \equiv \mathsf{f}_{n_1 n_2 \ldots n_a} \equiv \frac{\partial^a f(\underline{ж})}{\partial ж_{n_1} \partial ж_{n_2} \ldots \partial ж_{n_a}}$$

from which it follows in particular $\mathsf{f}_n \equiv \partial f(\underline{ж})/\partial ж_n$.

From: $o \geq 1$; (29); þ; Æ$\langle \tau_n, \mathsf{f}_n, o-1 \,/\!/\, \mathtt{f}, \mathtt{g}, h \,/\!/\, (30)\rangle$; follows

$$\mathsf{f}^{(o)} = (\mathsf{f}')^{(o-1)} = (\sum_{n=1}^{\hat{n}} \mathsf{f}_n \tau_n)^{(o-1)} = \sum_{n=1}^{\hat{n}} (\tau_n \mathsf{f}_n)^{(o-1)} = \sum_{n=1}^{\hat{n}} \sum_{k=0}^{o-1} \binom{o-1}{k} \tau_n^{(o-k-1)} \mathsf{f}_n^{(k)} \tag{31}$$

which gives rise to

$$|\mathsf{f}^{(o)}| \leq \sum_{n=1}^{\hat{n}} \sum_{k=0}^{o-1} \binom{o-1}{k} \left|\tau_n^{(o-k-1)} \mathsf{f}_n^{(k)}\right| \leq \Psi_o \hat{\Phi}_o \tag{32}$$

of which

$$\Psi_o \equiv \sum_{n=1}^{\hat{n}} \sum_{k=0}^{o-1} \binom{o-1}{k} |\tau_n^{(o-k-1)}| \qquad\qquad \hat{\Phi}_o \equiv \max\{|\mathsf{f}_n^{(k)}|; k = 0, o-1; n = 1, \hat{n}\}$$

Is placed $\mathcal{P}_\tau \equiv \{\tau_n(\mathsf{c}) \equiv \mathrm{T}_n; n = 1, \hat{\mathrm{n}}\}$ for which is had $\mathcal{P}_\tau$ if the $\{\tau_n; n = 1, \hat{\mathrm{n}}\}$ are constant as happens when the curve $\underline{\mathsf{c}}$ is a straight line segment. From: $\mathcal{P}_\tau$, first and fourth member of (31); follows IPM

$$\left\{|\mathsf{f}^{(o)}| = \left|\sum_{n=1}^{\hat{\mathrm{n}}} \tau_n \mathsf{f}_n^{(o-1)}\right| \le \sum_{n=1}^{\hat{\mathrm{n}}} |\tau_n \mathsf{f}_n^{(o-1)}| \le \Psi_{\mathrm{R}} |\mathsf{f}_{n_o}^{(o-1)}|\right\} \Leftarrow \mathcal{P}_\tau \tag{33}$$

of which

$$\Psi_{\mathrm{R}} \equiv \sum_{n=1}^{\hat{\mathrm{n}}} |\tau_n| \qquad\qquad |\mathsf{f}_{n_o}^{(o-1)}| \equiv \max\{|\mathsf{f}_n^{(o-1)}|; n = 1, \hat{\mathrm{n}}\}$$

The $\Psi_{\mathrm{R}} \le \Psi_o$, $|\mathsf{f}_{n_o}^{(o-1)}| \le \hat{\Phi}_o$, (32) and (33) show how $\mathcal{P}_\tau$ entails an upper bound of $|\mathsf{f}^{(o)}|$, i.e. of the maximum absolute value of a derivative defined on a curve, generally less than that implied by $\neg\mathcal{P}_\tau$.

From: $\mathcal{P}_\tau$, (31); $\mathcal{P}_\tau$, Æ$\langle \mathsf{f}_{n_1}, o-1 \mathbin{/\!/} \mathsf{f}^{(o)}, o \mathbin{/\!/} (31)\rangle$;. . . ; follows IPM

$$\left\{\mathsf{f}^{(o)} = \sum_{n_1=1}^{\hat{\mathrm{n}}} \tau_{n_1} \mathsf{f}_{n_1}^{(o-1)} = \sum_{n_1=1}^{\hat{\mathrm{n}}} \sum_{n_2=1}^{\hat{\mathrm{n}}} \tau_{n_1}\tau_{n_2} \mathsf{f}_{n_1 n_2}^{(o-2)} = \cdots = \sum_{n_1=1}^{\hat{\mathrm{n}}} \sum_{n_2=1}^{\hat{\mathrm{n}}} \cdots \sum_{n_o=1}^{\hat{\mathrm{n}}} \Theta_{n_1 n_2 \ldots n_o}\right\} \Leftarrow \mathcal{P}_\tau \tag{34}$$

of which $\Theta_{n_1 n_2 \ldots n_o} \equiv \tau_{n_1}\tau_{n_2} \ldots \tau_{n_o} \mathsf{f}_{n_1 n_2 \ldots n_o}$.

The numerosity of the set of all dispositions with repetition of class K of N objects is N^{K}. A m_{cba} is a-th element of b-th disposition with repetition of class c of $\{m = \check{m}, \hat{m}\}$. Is had

$$\sum_{m_1=\check{m}}^{\hat{m}} \sum_{m_2=\check{m}}^{\hat{m}} \cdots \sum_{m_{\mathrm{K}}=\check{m}}^{\hat{m}} \mathrm{G}_{m_1 m_2 \ldots m_{\mathrm{K}}} = \sum_{b=1}^{(\hat{m}-\check{m}+1)^{\mathrm{K}}} \mathrm{G}_{\{m_{\mathrm{K}ba}; a=1,\mathrm{K}\}} \tag{35}$$

The

$$\text{Æ}\langle \{n = 1, \hat{\mathrm{n}}\}, o, \Theta_{n_1 n_2 \ldots n_o} \mathbin{/\!/} \{m = \check{m}, \hat{m}\}, \mathrm{K}, \mathrm{G}_{m_1 m_2 \ldots m_{\mathrm{K}}} \mathbin{/\!/} (35)\rangle$$

entails

$$\sum_{n_1=1}^{\hat{\mathrm{n}}} \sum_{n_2=1}^{\hat{\mathrm{n}}} \cdots \sum_{n_o=1}^{\hat{\mathrm{n}}} \Theta_{n_1 n_2 \ldots n_o} = \sum_{b=1}^{\hat{\mathrm{n}}^o} \Theta_{\{n_{oba}; a=1,o\}} \tag{36}$$

where n_{oba} is a-th element of b-th disposition with repetition of class o of $\{n = 1, \hat{\mathrm{n}}\}$.

From: $\mathcal{P}_\tau$, (34), (36); expression of $\Theta_{n_1 n_2 \ldots n_o}$; follows IPM

$$\left\{|\mathsf{f}^{(o)}| = \left|\sum_{b=1}^{\hat{\mathrm{n}}^o} \Theta_{\{n_{oba}; a=1,o\}}\right| = \left|\sum_{b=1}^{\hat{\mathrm{n}}^o} \tau_{n_{ob1}}\tau_{n_{ob2}} \ldots \tau_{n_{obo}} \mathsf{f}_{n_{ob1} n_{ob2} \ldots n_{obo}}\right| \le \right.$$

$$\left. \sum_{b=1}^{\hat{\mathrm{n}}^o} |\tau_{n_{ob1}}\tau_{n_{ob2}} \ldots \tau_{n_{obo}} \mathsf{f}_{n_{ob1} n_{ob2} \ldots n_{obo}}| \le \Psi_{\mathrm{R}o} \hat{\Phi}_{\mathrm{R}o} \right\} \Leftarrow \mathcal{P}_\tau \tag{37}$$

of which

$$\Psi_{\mathrm{R}o} \equiv \sum_{b=1}^{\hat{n}^o} \breve{\Theta}_{\{n_{oba};a=1,o\}} \qquad \breve{\Theta}_{\{n_a;a=1,o\}} \equiv |\tau_{n_1}\tau_{n_2}\ldots\tau_{n_o}|$$

$$\hat{\Phi}_{\mathrm{R}o} \equiv \max\left\{\left|\frac{\partial^o f(\underline{ж})}{\partial ж_{n_1}\partial ж_{n_2}\ldots\partial ж_{n_o}}\right| /\!\!/ \{n_a \in \{n=1,\hat{n}\}; a=1,o\}\right\}$$

For the $|\mathsf{f}^{(o-1)}_{n_o}|$ of (33) is had $Æ\langle o-1, \mathsf{f}_{n_o} /\!\!/ o, \mathsf{f} /\!\!/ (33)\rangle$ which implies

$$\mathcal{P}_\tau \Rightarrow \left\{|\mathsf{f}^{(o-1)}_{n_o}| \le \Psi_{\mathrm{R}}\, |\mathsf{f}^{(o-2)}_{n_o n_{o-1}}|\right\} \qquad |\mathsf{f}^{(o-2)}_{n_o n_{o-1}}| \equiv \max\{\mathsf{f}^{(o-2)}_{n_o n}; n=1,\hat{n}\}$$

This, $\mathcal{P}_\tau$ and (33) entail $|\mathsf{f}^{(o)}| \le \Psi^2_{\mathrm{R}}\, |\mathsf{f}^{(o-2)}_{n_o n_{o-1}}|$; and, by decreasing subsequently in the same way up to 0 the derivation order of the second member, is reached the first bound of

$$\left\{|\mathsf{f}^{(o)}| \le \Psi^o_{\mathrm{R}}\, |\mathsf{f}_{n_o n_{o-1}\ldots n_1}| \le \Psi^o_{\mathrm{R}}\hat{\Phi}_{\mathrm{R}o}\right\} \Leftarrow \mathcal{P}_\tau \tag{38}$$

of which $|\mathsf{f}_{n_o n_{o-1}\ldots n_1}| \equiv \max\{|\mathsf{f}_{n_o n_{o-1}\ldots n}|; n=1,\hat{n}\}$.

The comparison between (37) and (38) show $\Psi_{\mathrm{R}o} = \Psi^o_{\mathrm{R}}$ which can be confirmed as follows. From: definition of $\Psi_{\mathrm{R}o}$;

$$Æ\langle\{n=1,\hat{n}\}, o, \breve{\Theta}_{\{n_{oba};a=1,o\}} /\!\!/ \{m=\check{m},\hat{m}\}, \mathrm{K}, \mathrm{G}_{\{m_{\mathrm{K}ba};a=1,\mathrm{K}\}} /\!\!/ (35)\rangle;$$

definition of $\breve{\Theta}_{\{n_a;a=1,o\}}$; þ; þ; definition of Ψ_{R}; follows

$$\Psi_{\mathrm{R}o} \equiv \sum_{b=1}^{\hat{n}^o} \breve{\Theta}_{\{n_{oba};a=1,o\}} = \sum_{n_1=1}^{\hat{n}}\sum_{n_2=1}^{\hat{n}}\cdots\sum_{n_o=1}^{\hat{n}} \breve{\Theta}_{n_1 n_2\ldots n_o} = \sum_{n_1=1}^{\hat{n}}\sum_{n_2=1}^{\hat{n}}\cdots\sum_{n_o=1}^{\hat{n}} |\tau_{n_1}\tau_{n_2}\ldots\tau_{n_o}| =$$

$$\sum_{n_1=1}^{\hat{n}}|\tau_{n_1}|\sum_{n_2=1}^{\hat{n}}|\tau_{n_2}|\ldots\sum_{n_o=1}^{\hat{n}}|\tau_{n_o}| = \Big(\sum_{n=1}^{\hat{n}}|\tau_n|\Big)^o = \Psi^o_{\mathrm{R}}$$

4 The approximation of a linear combination of directional derivatives that expresses a partial derivative at a intersection of some curves

Is considered the set of curves $\{\underline{\mathsf{C}}_c; c=1,\hat{c}\}$ of which $\hat{c} \ge \hat{n}$,

$$\{Æ\langle\underline{\mathsf{C}}_c, \mathsf{c}_c, \underline{ж}_c, ж_{cn}, \check{\mathsf{c}}_c, \hat{\mathsf{c}}_c /\!\!/ \underline{\mathsf{C}}, \mathsf{c}, \underline{ж}, ж_n, \check{\mathsf{c}}, \hat{\mathsf{c}} /\!\!/ (29)\rangle, \underline{\mathrm{x}} = \underline{ж}_c(\tilde{\mathsf{c}}_c); c=1,\hat{c}\}$$

$$\mathrm{C}_{c1} = \check{\mathsf{c}}_c = 0 \qquad \mathrm{C}_{c\hat{\mathrm{i}}_c} = \hat{\mathsf{c}}_c \qquad \{\mathrm{C}_{c,i-1} < \mathrm{C}_{ci}; i=2,\hat{\mathrm{i}}_c\} \qquad \tilde{\mathsf{c}}_c \equiv \mathrm{C}_{c\mathrm{I}_c} \qquad \mathrm{I}_c \in \{i=1,\hat{\mathrm{i}}_c\}$$

This gives rise to the linear system $\underline{\underline{\mathsf{T}}} \cdot \underline{\mathsf{D}} = \underline{\mathsf{F}}$ of which

$$\underline{\underline{\mathsf{T}}} \equiv [\tau_{cn}; c=1,\hat{c}; n=1,\hat{n}] \qquad \tau_{cn} = ж'_{cn}(\tilde{\mathsf{c}}_c) \qquad \underline{\mathsf{D}} \equiv \{\frac{\partial f(\underline{\mathrm{x}})}{\partial \mathrm{x}_n}; n=1,\hat{n}\}$$

$$\{\frac{\partial f(\underline{\mathrm{x}})}{\partial \mathrm{x}_n} = \frac{\partial f(\underline{ж}_c(\tilde{\mathsf{c}}_c))}{\partial ж_{cn}}; c=1,\hat{c}\} \qquad \underline{\mathsf{F}} \equiv \{\mathsf{F}_c; c=1,\hat{c}\} \qquad \mathsf{F}_c = \mathsf{f}'_c(\tilde{\mathsf{c}}_c) \qquad \mathsf{f}_c(\mathsf{c}_c) \equiv f(\underline{ж}_c(\mathsf{c}_c))$$

Is considered the $\{c_{nb}; n = 1, \hat{\mathrm{n}}; b = 1, \hat{\mathrm{b}}\}$ that verifies

$$\{\det \underline{\underline{\mathsf{T}}}_b \neq 0; b = 1, \hat{\mathrm{b}}\} \qquad \underline{\underline{\mathsf{T}}}_b \equiv [\tau_{c_{nb}\mathrm{n}}; n = 1, \hat{\mathrm{n}}; \mathrm{n} = 1, \hat{\mathrm{n}}]$$

This and (2.3.4) of [1] entail $\{\underline{\mathsf{D}} = \underline{\underline{\mathsf{T}}}_b^{-1} \cdot \underline{\mathsf{F}}_b; b = 1, \hat{\mathrm{b}}\}$ of which

$$\underline{\underline{\mathsf{T}}}_b^{-1} \equiv [\mathcal{T}_{bn\mathrm{n}}; n = 1, \hat{\mathrm{n}}; \mathrm{n} = 1, \hat{\mathrm{n}}] \qquad \underline{\mathsf{F}}_b \equiv \{\mathsf{F}_{c_{nb}}; n = 1, \hat{\mathrm{n}}\}$$

and which gives rise to

$$\{\underline{\mathsf{D}}_{\mathrm{A}b} = \underline{\underline{\mathsf{T}}}_b^{-1} \cdot \underline{\mathsf{F}}_{\mathrm{A}b}; b = 1, \hat{\mathrm{b}}\} \qquad \underline{\mathsf{D}}_{\mathrm{A}b} \equiv \{-\mathsf{D}_{\mathrm{A}bn}; n = 1, \hat{\mathrm{n}}\} \qquad \underline{\mathsf{F}}_{\mathrm{A}b} \equiv \{-\mathsf{F}_{\mathrm{A}c_{nb}}; n = 1, \hat{\mathrm{n}}\}$$

The $\underline{\mathsf{D}} = \underline{\underline{\mathsf{T}}}_b^{-1} \cdot \underline{\mathsf{F}}_b$ and $\underline{\mathsf{D}}_{\mathrm{A}b} = \underline{\underline{\mathsf{T}}}_b^{-1} \cdot \underline{\mathsf{F}}_{\mathrm{A}b}$ imply $\underline{\mathsf{D}} + \underline{\mathsf{D}}_{\mathrm{A}b} = \underline{\underline{\mathsf{T}}}_b^{-1} \cdot \underline{\mathsf{F}}_b + \underline{\underline{\mathsf{T}}}_b^{-1} \cdot \underline{\mathsf{F}}_{\mathrm{A}b}$ from which follows

$$\underline{\tilde{\mathcal{E}}}_b = \underline{\underline{\mathsf{T}}}_b^{-1} \cdot \underline{\mathcal{E}}_b \qquad \underline{\tilde{\mathcal{E}}}_b \equiv \underline{\mathsf{D}} + \underline{\mathsf{D}}_{\mathrm{A}b} \equiv \{\tilde{\mathcal{E}}_{bn}; n = 1, \hat{\mathrm{n}}\} \qquad \tilde{\mathcal{E}}_{bn} = \frac{\partial f(\underline{\mathrm{x}})}{\partial \mathrm{x}_n} - \mathsf{D}_{\mathrm{A}bn}$$

$$\underline{\mathcal{E}}_b \equiv \underline{\mathsf{F}}_b + \underline{\mathsf{F}}_{\mathrm{A}b} \equiv \{\mathcal{E}_{c_{nb}}; n = 1, \hat{\mathrm{n}}\} \qquad \mathcal{E}_c = \mathsf{F}_c - \mathsf{F}_{\mathrm{A}c}$$

Is considered every $\mathsf{F}_{\mathrm{A}c}$ as a known approximation of the unknown F_c, by following, for $\underline{\mathsf{D}}_{\mathrm{A}b} = \underline{\underline{\mathsf{T}}}_b^{-1} \cdot \underline{\mathsf{F}}_{\mathrm{A}b}$ and $\underline{\mathsf{D}} = \underline{\underline{\mathsf{T}}}_b^{-1} \cdot \underline{\mathsf{F}}_b$, that every $\mathsf{D}_{\mathrm{A}bn}$ is a known approximation of the unknown $\partial f / \partial \mathrm{x}_n$, and therefore that $\mathcal{E}_{c_{nb}}$ and $\tilde{\mathcal{E}}_{bn}$ are the errors of the respective approximations of $\mathsf{F}_{c_{nb}}$ with $\mathsf{F}_{\mathrm{A}c_{nb}}$ and of $\partial f / \partial \mathrm{x}_n$ with $\mathsf{D}_{\mathrm{A}bn}$.

The $Æ\langle \mathrm{C}_{ci}, \mathsf{f}_c(\mathrm{C}_{ci}); i = 1, \hat{\mathrm{i}}_c \,/\!/\, \mathsf{x}_p, \mathsf{y}(\mathsf{x}_p); p = 1, \hat{\mathrm{p}} \,/\!/\, (14)\rangle$ allows to place

$$\mathsf{F}_{\mathrm{A}c} = \sum_{i=1}^{\hat{\mathrm{i}}_c} \Lambda_{ci} \mathsf{f}_c(\mathrm{C}_{ci}) \tag{39}$$

whose $\{\Lambda_{ci}; i = 1, \hat{\mathrm{i}}_c\}$ are knowable by means of

$$Æ\langle \tilde{\mathsf{c}}_c, \{\mathrm{C}_{ci}; i = 1, \hat{\mathrm{i}}_c\} \,/\!/\, \mathsf{x}_p, \{\mathsf{x}_{\mathrm{p}}; \mathrm{p} = 1, \hat{\mathrm{p}}\} \,/\!/\, (14)\rangle$$

$$\{Æ\langle \Lambda_{ci}; i = 1, \hat{\mathrm{i}}_c \,/\!/\, \lambda_{\mathrm{pp}}; \mathrm{p} = 1, \hat{\mathrm{p}} \,/\!/\, (14)\rangle; \forall \mathrm{I}_c < \hat{\mathrm{i}}_c\}$$

$$\{Æ\langle \Lambda_{ci}; i = 1, \hat{\mathrm{i}}_c \,/\!/\, \vartheta_{\mathrm{p}}; \mathrm{p} = 1, \hat{\mathrm{p}} \,/\!/\, (14)\rangle; \forall \mathrm{I}_c = \hat{\mathrm{i}}_c\}$$

The $Æ\langle \mathsf{F}_c, \mathsf{F}_{\mathrm{A}c}, -\mathcal{E}_c \,/\!/\, \mathsf{y}'(\mathsf{x}_p), \tilde{\mathrm{S}}'(\mathsf{x}_p), \mathcal{E}'_p \,/\!/\, \text{section 2.3}\rangle$, due to (39) and $\mathcal{E}_c = \mathsf{F}_c - \mathsf{F}_{\mathrm{A}c}$, entails $|\mathcal{E}_c| \leq \mathcal{K}_c \varphi_c$ whose $\mathcal{K}_c$ is knowable by means of $Æ\langle -\mathcal{E}_c, \mathcal{K}_c, \varphi_c \,/\!/\, \mathcal{E}'_p, \mathrm{K}_p, \Phi\langle \mathsf{y}^{(4)}\rangle \,/\!/\, (28)\rangle$ and of which $\varphi_c \equiv \max\{|\mathsf{f}_c^{(4)}| \,/\!/\, \mathsf{c} \in (0, \hat{\mathsf{c}}_c)\}$.

This and $Æ\langle \mathsf{f}_c(\mathsf{c}_c), 4 \,/\!/\, \mathsf{f}(\mathsf{c}), o \,/\!/\, (32), (38)\rangle$ allow to place $\varphi_c = \breve{\psi}_c \phi_c$ whose $\breve{\psi}_c$ is known. Therefore is had $|\mathcal{E}_c| \leq \psi_c \phi_c$ of which $\psi_c \equiv \mathcal{K}_c \breve{\psi}_c$ and with ψ_c known.

The $\underline{\tilde{\mathcal{E}}}_b = \underline{\underline{\mathsf{T}}}_b^{-1} \cdot \underline{\mathcal{E}}_b$ has the expression $\{\tilde{\mathcal{E}}_{bn} = \sum_{\mathrm{n}=1}^{\hat{\mathrm{n}}} \mathcal{T}_{bn\mathrm{n}} \mathcal{E}_{c_{\mathrm{n}b}}; n = 1, \hat{\mathrm{n}}\}$ that, on the basis of $|\mathcal{E}_c| \leq \psi_c \phi_c$, gives rise to $\{|\tilde{\mathcal{E}}_{bn}| \leq \breve{\psi}_{bn} \hat{\phi}; n = 1, \hat{\mathrm{n}}\}$ of which $\hat{\phi} \equiv \max\{\phi_c; c = 1, \hat{\mathrm{c}}\}$ and whose $\breve{\psi}_{bn}$ is made known by $\breve{\psi}_{bn} \equiv \sum_{\mathrm{n}=1}^{\hat{\mathrm{n}}} |\mathcal{T}_{bn\mathrm{n}}| \psi_{c_{\mathrm{n}b}}$.

These bounds imply that the most convenient, among the alternative $\{\tilde{\mathcal{E}}_{bn}; b = 1, \hat{\mathrm{b}}\}$, is the $\tilde{\mathcal{E}}_{\tilde{\mathrm{B}}n}$ of which $\tilde{\mathrm{B}} \equiv \{b \,\|\, \breve{\psi}_{bn} = \min\{\breve{\psi}_{\mathrm{b}n}; \mathrm{b} = 1, \hat{\mathrm{b}}\}\}$.

From: this and $\tilde{\mathcal{E}}_{bn} \equiv \partial f/\partial \mathrm{x}_n - \mathsf{D}_{\mathrm{A}bn}$; $\mathsf{D}_{\mathrm{A}bn} = \sum_{\mathrm{n}=1}^{\hat{\mathrm{n}}} \mathsf{T}_{bn\mathrm{n}} \mathsf{F}_{\mathrm{A}c_{\mathrm{n}b}}$ (due to $\underline{\mathsf{D}}_{\mathrm{A}b} = \underline{\underline{\mathsf{T}}}_b^{-1} \cdot \underline{\mathsf{F}}_{\mathrm{A}b}$); (39); $\mathsf{f}_c(\mathrm{C}_c) \equiv f(\underline{\text{ж}}_c(\mathrm{C}_c))$; follows

$$\frac{\partial f(\underline{\mathrm{x}})}{\partial \mathrm{x}_n} = \mathsf{D}_{\mathrm{A}\tilde{\mathrm{B}}n} + \tilde{\mathcal{E}}_{\tilde{\mathrm{B}}n} = \sum_{\mathrm{n}=1}^{\hat{\mathrm{n}}} \mathsf{T}_{\tilde{\mathrm{B}}n\mathrm{n}} \mathsf{F}_{\mathrm{A}c_{\mathrm{n}\tilde{\mathrm{B}}}} + \tilde{\mathcal{E}}_{\tilde{\mathrm{B}}n} = \sum_{\mathrm{n}=1}^{\hat{\mathrm{n}}} \mathsf{T}_{\tilde{\mathrm{B}}n\mathrm{n}} \sum_{i=1}^{\hat{\mathrm{i}}_{c_{\mathrm{n}\tilde{\mathrm{B}}}}} \Lambda_{c_{\mathrm{n}\tilde{\mathrm{B}}}i} \mathsf{f}_{c_{\mathrm{n}\tilde{\mathrm{B}}}}(\mathrm{C}_{c_{\mathrm{n}\tilde{\mathrm{B}}}i}) + \tilde{\mathcal{E}}_{\tilde{\mathrm{B}}n} =$$
$$\sum_{\mathrm{n}=1}^{\hat{\mathrm{n}}} \mathsf{T}_{\tilde{\mathrm{B}}n\mathrm{n}} \sum_{i=1}^{\hat{\mathrm{i}}_{c_{\mathrm{n}\tilde{\mathrm{B}}}}} \Lambda_{c_{\mathrm{n}\tilde{\mathrm{B}}}i} f(\underline{\text{ж}}_{c_{\mathrm{n}\tilde{\mathrm{B}}}}(\mathrm{C}_{c_{\mathrm{n}\tilde{\mathrm{B}}}i})) + \tilde{\mathcal{E}}_{\tilde{\mathrm{B}}n}$$

that, admitting $\tilde{\mathcal{E}}_{\tilde{\mathrm{B}}n} \approx 0$, gives rise to

$$\frac{\partial f(\underline{\mathrm{x}})}{\partial \mathrm{x}_n} \approx \sum_{\mathrm{n}=1}^{\hat{\mathrm{n}}} \sum_{i=1}^{\hat{\mathrm{i}}_{c_{\mathrm{n}\tilde{\mathrm{B}}}}} \mathsf{T}_{\tilde{\mathrm{B}}n\mathrm{n}} \Lambda_{c_{\mathrm{n}\tilde{\mathrm{B}}}i} \mathsf{f}_{c_{\mathrm{n}\tilde{\mathrm{B}}}}(\mathrm{C}_{c_{\mathrm{n}\tilde{\mathrm{B}}}i}) \tag{40}$$

as an approximation of the linear combination of directional derivatives $\sum_{\mathrm{n}=1}^{\hat{\mathrm{n}}} \mathsf{T}_{\tilde{\mathrm{B}}n\mathrm{n}} \mathsf{F}_{c_{\mathrm{n}\tilde{\mathrm{B}}}}$, which, according to $\underline{\mathsf{D}} = \underline{\underline{\mathsf{T}}}_{\tilde{\mathrm{B}}}^{-1} \cdot \underline{\mathsf{F}}_{\tilde{\mathrm{B}}}$, expresses $\partial f(\underline{\mathrm{x}})/\partial \mathrm{x}_n$ at the point $\underline{\mathrm{x}}$ of which $\{\underline{\mathrm{x}} = \underline{\text{ж}}_c(\tilde{\mathsf{C}}_c); c = 1, \hat{\mathrm{c}}\}$.

5 The formulation of a differential analytical model and its numerical solution as the unknowns of a total system

The generic differential analytical model $\mathfrak{M}$, whose solution is (as said in the introduction) the purpose of this paper, has an expression $\mathfrak{M}\langle\underline{\mathrm{x}}\rangle$ of which

$$\mathfrak{M}_{\underline{\mathrm{x}}} \equiv \{\mathrm{E}_m\big(\underline{\mathrm{F}}_m(\underline{\mathrm{x}}), \underline{\mathrm{D}}_m(\underline{\mathrm{x}}), \underline{\mathrm{F}}_{\mathrm{N}m}(\underline{\mathrm{x}})\big) = 0; m = 1, \hat{\mathfrak{m}}\} \qquad \underline{\mathrm{F}}_m(\underline{\mathrm{x}}) \subseteq \underline{\mathrm{F}}(\underline{\mathrm{x}}) \qquad \underline{\mathrm{D}}_m(\underline{\mathrm{x}}) \subseteq \underline{\mathrm{D}}(\underline{\mathrm{x}})$$
$$\underline{\mathrm{F}}_{\mathrm{N}m}(\underline{\mathrm{x}}) \subseteq \underline{\mathrm{F}}_{\mathrm{N}}(\underline{\mathrm{x}}) \qquad \underline{\mathrm{F}}(\underline{\mathrm{x}}) \equiv \{\mathrm{F}_m(\underline{\mathrm{x}}); m = 1, \hat{\mathrm{m}}\} \qquad \underline{\mathrm{D}}(\underline{\mathrm{x}}) \equiv \{\mathrm{D}_d(\underline{\mathrm{x}}); d = 1, \hat{\mathrm{d}}\}$$
$$\underline{\mathrm{F}}_{\mathrm{N}}(\underline{\mathrm{x}}) \equiv \{\mathrm{F}_{\mathrm{N}m}(\underline{\mathrm{x}}); m = \hat{\mathrm{m}} + 1, \hat{\mathrm{M}}\} \qquad \underline{\bar{\mathrm{F}}}_d(\underline{\mathrm{x}}) \equiv \{\bar{\mathrm{F}}_{do}(\underline{\mathrm{x}}); o = 0, \hat{\mathrm{O}}_d\}$$
$$\hat{\mathrm{m}} \le \hat{\mathfrak{m}} \qquad \hat{\mathrm{O}}_d > 0 \qquad m_d \in \{m = 1, \hat{\mathrm{m}}\} \qquad \bar{\mathrm{F}}_{d0} \equiv \mathrm{F}_{m_d} \qquad \bar{\mathrm{F}}_{d\hat{\mathrm{O}}_d} \equiv \mathrm{D}_d$$
$$\mathrm{D}_d \equiv \frac{\partial^{\hat{\mathrm{O}}_d} \mathrm{F}_{m_d}}{\partial \mathrm{x}_{n_{d1}} \partial \mathrm{x}_{n_{d2}} \ldots \partial \mathrm{x}_{n_{d\hat{\mathrm{O}}_d}}} \qquad \{\bar{\mathrm{F}}_{do} \equiv \frac{\partial \bar{\mathrm{F}}_{d,o-1}}{\partial \mathrm{x}_{n_{do}}} \equiv \frac{\partial^o \mathrm{F}_{m_d}}{\partial \mathrm{x}_{n_{d1}} \partial \mathrm{x}_{n_{d2}} \ldots \partial \mathrm{x}_{n_{do}}}; o = 1, \hat{\mathrm{O}}_d\}$$

where: $\underline{\mathrm{F}}(\underline{\mathrm{x}})$ are unknown functions which appear autonomously (i.e. as derivatives of order 0) or in $\underline{\mathrm{D}}(\underline{\mathrm{x}})$ as functions to be derived; $\underline{\mathrm{D}}(\underline{\mathrm{x}})$ are unknown functions, since they are derivatives whose functions to be derived are the unknown $\underline{\mathrm{F}}(\underline{\mathrm{x}})$; $\underline{\mathrm{F}}_{\mathrm{N}}(\underline{\mathrm{x}})$ are functions whose values are known at the points of $\mathfrak{R}\langle\underline{\mathrm{x}}\rangle$ where is desired to know the $\underline{\mathrm{F}}(\underline{\mathrm{x}})$; each $\{n_{d\mathrm{o}}; \mathrm{o} = 1, o\}$ (of which $1 \le o \le \hat{\mathrm{O}}_d$) is a combination generally with repetition of class o of the $\{n = 1, \hat{\mathrm{n}}\}$ and is replaceable by an its permutation as enables the Schwarz's theorem ([10],[11]); $\underline{\mathrm{x}}$ is arbitrary unless $\underline{\mathrm{x}} \in \mathfrak{R}_{\underline{\mathrm{x}}}$.

The knowledge of $\underline{\mathrm{F}}(\underline{\mathrm{x}})$ can be considered prevented by not knowing $\underline{\mathrm{D}}(\underline{\mathrm{x}})$, in the sense that, if these were functions known of $\underline{\mathrm{F}}(\underline{\mathrm{x}})$, $\mathfrak{M}\langle\underline{\mathrm{x}}\rangle$ would be solved as a system of $\hat{\mathfrak{m}}$ non differential equations in the $\hat{\mathrm{m}}$ unknowns constituted by the $\underline{\mathrm{F}}(\underline{\mathrm{x}})$. Therefore a solution of $\mathfrak{M}$, numerical as it is understood in the following, can be obtained by removing this obstacle by means of approximating the values of $\underline{\mathrm{D}}(\underline{\mathrm{x}})$ with known functions of values of $\underline{\mathrm{F}}(\underline{\mathrm{x}})$.

One such solution is a set of numbers called $\underline{\mathfrak{F}}$, of which

$$\underline{\mathfrak{F}} \equiv \{\mathfrak{f}_{mp}; m = 1, \hat{\mathrm{m}}; p = 1, \hat{\mathrm{P}}\} \qquad \mathfrak{f}_{mp} \approx \mathrm{F}_m(\underline{\mathrm{x}}_p) \qquad \underline{\mathrm{x}}_p \in \mathfrak{R}_{\underline{\mathrm{x}}}$$
$$\underline{\mathfrak{X}} \equiv \{\underline{\mathrm{x}}_p; p = 1, \hat{\mathrm{P}}\} \subset \mathfrak{R}_{\underline{\mathrm{x}}} \qquad 1 \leq \hat{\mathrm{P}} \neq \infty$$

In seeking a $\underline{\mathfrak{F}}$, are given as known $\mathfrak{R}_{\underline{\mathrm{x}}}$, $\underline{\mathfrak{X}}$, $\{\underline{\mathrm{F}}_{\mathrm{N}}(\underline{\mathrm{x}}_p); p = 1, \hat{\mathrm{P}}\}$ and contingently a set of conditions $\underline{\mathfrak{C}}$ of which

$$\underline{\mathfrak{C}} \subset \Big\{\{\mathrm{F}_m(\underline{\mathrm{x}}_p) - \tilde{\mathrm{F}}_{mp} = 0; m = 1, \hat{\mathrm{m}}\},$$
$$\{\bar{\mathrm{F}}_{do}(\underline{\mathrm{x}}_p) - \bar{\mathbf{F}}_{dop} = 0; o = 1, \hat{\mathrm{O}}_d; d = 1, \hat{\mathrm{d}}\}; p = 1, \hat{\mathrm{P}}\Big\} \quad (41)$$

where each $\mathrm{F}_m(\underline{\mathrm{x}}_p) - \tilde{\mathrm{F}}_{mp} = 0$ means that $\mathrm{F}_m(\underline{\mathrm{x}})$ has at $\underline{\mathrm{x}}_p$ the known value $\tilde{\mathrm{F}}_{mp}$, and similarly for each $\bar{\mathrm{F}}_{do}(\underline{\mathrm{x}}_p) - \bar{\mathbf{F}}_{dop} = 0$.

These $\underline{\mathfrak{C}}$ imply the eventuality that at some $\underline{\mathrm{x}}_p$ become all known the $\underline{\mathrm{F}}(\underline{\mathrm{x}}_p)$ i.e. the chance of some relations of type

$$\{\mathrm{F}_m(\underline{\mathrm{x}}_p) - \tilde{\mathrm{F}}_{mp} = 0; m = 1, \hat{\mathrm{m}}\} \subset \underline{\mathfrak{C}}$$

and therefore give rise to a

$$\underline{\tilde{\mathfrak{X}}} \equiv \{\underline{\tilde{\mathrm{x}}}_{\mathrm{p}}; \mathrm{p} = 1, \hat{\mathfrak{p}}\} \equiv \{\underline{\mathrm{x}}_{p_{\mathrm{p}}}; \mathrm{p} = 1, \hat{\mathfrak{p}}\} \subseteq \underline{\mathfrak{X}}$$

such that the numerosity of $\underline{\tilde{\mathfrak{X}}}$ is the maximum compatibly with being at each $\underline{\tilde{\mathrm{x}}}_{\mathrm{p}}$ unknown at least one of the $\underline{\mathrm{F}}(\underline{\tilde{\mathrm{x}}}_{\mathrm{p}})$. Moreover is had also the eventuality that the $\underline{\mathfrak{C}}$ do take the form $0 = 0$ to some equations of a $\mathfrak{M}\langle\underline{\tilde{\mathrm{x}}}_{\mathrm{p}}\rangle$.

Is called $\mathfrak{M}_{\mathrm{p}}$ the system of $\hat{\mathfrak{m}}_{\mathrm{p}}$ equations constituted by the

$$\underline{\mathfrak{C}} \cap \{\bar{\mathrm{F}}_{do}(\underline{\mathrm{x}}_{p_{\mathrm{p}}}) - \bar{\mathbf{F}}_{dop_{\mathrm{p}}} = 0; o = 1, \hat{\mathrm{O}}_d; d = 1, \hat{\mathrm{d}}\}$$

and by those that are obtained by removing from $\mathfrak{M}\langle\underline{\tilde{\mathrm{x}}}_{\mathrm{p}}\rangle$ the equations that the $\underline{\mathfrak{C}}$ reduce to the useless form $0 = 0$.

A such $\mathfrak{M}_{\mathrm{p}}$ can be lacking in some of the $\Big\{\{\mathrm{F}_m(\underline{\tilde{\mathrm{x}}}_{\mathrm{p}}); m = 1, \hat{\mathrm{m}}\}, \{\mathrm{D}_d(\underline{\tilde{\mathrm{x}}}_{\mathrm{p}}); d = 1, \hat{\mathrm{d}}\}\Big\}$, may include some of the $\underline{\mathfrak{C}}$ and is had

$$\mathfrak{M}_{\mathrm{p}} \equiv \{E_{\mathrm{pm}}\big(\underline{F}_{\mathrm{pm}}, \underline{D}_{\mathrm{pm}}(\underline{\tilde{\mathrm{x}}}_{\mathrm{p}})\big); \mathrm{m} = 1, \hat{\mathfrak{m}}_{\mathrm{p}}\} \qquad \underline{F}_{\mathrm{pm}} \subseteq \underline{F}_{\mathrm{p}} \equiv \{F_{\mathrm{pm}}; \mathrm{m} = 1, \tilde{\mathfrak{m}}_{\mathrm{p}}\} \subseteq \underline{\mathrm{F}}(\underline{\tilde{\mathrm{x}}}_{\mathrm{p}})$$
$$\underline{D}_{\mathrm{pm}}(\underline{\tilde{\mathrm{x}}}_{\mathrm{p}}) \subseteq \underline{D}_{\mathrm{p}}(\underline{\tilde{\mathrm{x}}}_{\mathrm{p}}) \equiv \{D_{\mathrm{pd}}(\underline{\tilde{\mathrm{x}}}_{\mathrm{p}}); \mathrm{d} = 1, \hat{\mathfrak{d}}_{\mathrm{p}}\} \qquad \tilde{\mathfrak{m}}_{\mathrm{p}} \leq \hat{\mathfrak{m}}_{\mathrm{p}} \qquad \ddot{\mathrm{O}}_{\mathrm{pd}} > 0 \qquad m_{\mathrm{pd}} \in \{m = 1, \hat{\mathrm{m}}\}$$
$$\underline{\ddot{\mathrm{F}}}_{\mathrm{pd}} \equiv \{\ddot{\mathrm{F}}_{\mathrm{pd}o}; o = 0, \ddot{\mathrm{O}}_{\mathrm{pd}}\} \qquad \ddot{\mathrm{F}}_{\mathrm{pd}0} \equiv \mathrm{F}_{m_{\mathrm{pd}}} \qquad \ddot{\mathrm{F}}_{\mathrm{pd}\ddot{\mathrm{O}}_{\mathrm{pd}}} \equiv D_{\mathrm{pd}}$$
$$D_{\mathrm{pd}} \equiv \frac{\partial^{\ddot{\mathrm{O}}_{\mathrm{pd}}} \mathrm{F}_{m_{\mathrm{pd}}}}{\partial \mathrm{x}_{n_{\mathrm{pd}1}} \partial \mathrm{x}_{n_{\mathrm{pd}2}} \ldots \partial \mathrm{x}_{n_{\mathrm{pd}\ddot{\mathrm{O}}_{\mathrm{pd}}}}}$$
$$\{\ddot{\mathrm{F}}_{\mathrm{pd}o} \equiv \frac{\partial \ddot{\mathrm{F}}_{\mathrm{pd},o-1}}{\partial \mathrm{x}_{n_{\mathrm{pd}o}}} \equiv \frac{\partial^o \mathrm{F}_{m_{\mathrm{pd}}}}{\partial \mathrm{x}_{n_{\mathrm{pd}1}} \partial \mathrm{x}_{n_{\mathrm{pd}2}} \ldots \partial \mathrm{x}_{n_{\mathrm{pd}o}}}; o = 1, \ddot{\mathrm{O}}_{\mathrm{pd}}\} \quad (42)$$

where: $\underline{F}_{\rm p}$ are unknown and appear autonomously or as functions to be derived; $\underline{D}_{\rm p}(\underline{\widetilde{\rm x}}_{\rm p})$ are all treated as unknown even if some of them are made known by $\underline{\mathfrak{C}}$; each $\{n_{{\rm pd}o}; o = 1, \ddot{\rm o}_{\rm pd}\}$ is a combination generally with repetition of class $\ddot{\rm o}_{\rm pd}$ of the $\{n = 1, \hat{\rm n}\}$.

Therefore the total system $\underline{\mathfrak{M}}$, of which $\underline{\mathfrak{M}} \equiv \{\mathfrak{M}_{\rm p}; {\rm p} = 1, \hat{\mathfrak{p}}\}$, is a system of $\mathfrak{N}_{\rm E}$ equations, of which $\mathfrak{N}_{\rm E} = \sum_{\rm p=1}^{\hat{\mathfrak{p}}} \hat{\mathfrak{m}}_{\rm p}$, where are unknown $\{\underline{D}_{\rm p}(\underline{\widetilde{\rm x}}_{\rm p}); {\rm p} = 1, \hat{\mathfrak{p}}\}$ and the $\mathfrak{N}_{\rm I}$ values $\{\underline{F}_{\rm p}; {\rm p} = 1, \hat{\mathfrak{p}}\}$ of which $\mathfrak{N}_{\rm I} = \sum_{\rm p=1}^{\hat{\mathfrak{p}}} \tilde{\mathfrak{m}}_{\rm p} \leq \mathfrak{N}_{\rm E}$. Expressing then each of $\{D_{\rm pd}(\underline{\widetilde{\rm x}}_{\rm p}); {\rm d} = 1, \hat{\mathfrak{d}}_{\rm p}; {\rm p} = 1, \hat{\mathfrak{p}}\}$ by means of a respective

$$D_{\rm pd}(\underline{\widetilde{\rm x}}_{\rm p}) = \tilde{\rm D}_{\rm pd}(\underline{\mathcal{F}}_{\rm pd}, \mathcal{E}_{\rm Tpd}) \equiv D(\underline{\mathcal{F}}_{\rm pd}) + \mathcal{E}_{\rm Tpd} \approx D(\underline{\mathcal{F}}_{\rm pd}) \tag{43}$$

of which $\underline{\mathcal{F}}_{\rm pd} \equiv \{\mathcal{F}_{{\rm pd}i}; i = 1, \hat{\mathfrak{p}}_{\rm pd}\} \subseteq \{{\rm F}_{m_{\rm pd}}(\underline{\rm x}_p); p = 1, \hat{\rm P}\}$, $\underline{\mathcal{F}}_{\rm pd} \subseteq \{\underline{F}_{\rm p}; {\rm p} = 1, \hat{\mathfrak{p}}\}$, and where $D(\underline{\mathcal{F}}_{\rm pd})$ is a known function that approximates $\tilde{\rm D}_{\rm pd}(\underline{\mathcal{F}}_{\rm pd}, \mathcal{E}_{\rm Tpd})$ with an error $\mathcal{E}_{\rm Tpd}$ of which is admitted $\mathcal{E}_{\rm Tpd} \approx 0$; $\underline{\mathfrak{M}}$ becomes a system $\{E_{{\rm T}s}(\underline{F}_{\rm p}; {\rm p} = 1, \hat{\mathfrak{p}}) = 0; s = 1, \mathfrak{N}_{\rm E}\}$, nondifferential and generally nonlinear, of $\mathfrak{N}_{\rm E}$ equations in the $\mathfrak{N}_{\rm I}$ unknowns $\{F_{\rm pm}; {\rm m} = 1, \tilde{\mathfrak{m}}_{\rm p}; {\rm p} = 1, \hat{\mathfrak{p}}\}$, and that according to $\mathfrak{N}_{\rm I} \leq \mathfrak{N}_{\rm E}$ can be solved with the known methods of numerical analysis such as that of Newton-Raphson reported in section 2.4.5 of [1].

Indeed this method is feasible by calculating some successive solutions of a linear system and, in the case, as is $\underline{\mathfrak{M}}$, of a number of equations ${\rm N_E}$ not less than the number of unknowns ${\rm N_I}$, each of these solutions is obtainable by applying the (2.3.4) of [1], and in particular using the Gauss's method with the variant of "maximum pivot" and with the only modification of considering all the ${\rm N_E}$ equations even if the implicated pivots are only ${\rm N_I}$.

In seeking a $\underline{\mathfrak{F}}$, as the $\underline{\mathfrak{C}}$ are contingently known other conditions, each consisting of an equation similar to those of $\mathfrak{M}$, i.e. of the type ${\rm A}(\underline{\rm B}(\underline{\rm x}), \underline{\rm C}(\underline{\rm x}), \underline{\rm x}) = 0$ of which $\underline{\rm B}(\underline{\rm x}) \subseteq \underline{\rm F}(\underline{\rm x})$ and $\underline{\rm C}(\underline{\rm x}) \subseteq \underline{\rm D}(\underline{\rm x})$, but imposed only on some of the $\underline{\mathfrak{X}}$ i.e. only on the elements of a $\underline{\dot{\mathfrak{X}}}$ of which $\underline{\mathfrak{X}} \supset \underline{\dot{\mathfrak{X}}}$. Each of these other conditions can be introduced as follows: are added to $\underline{\rm F}(\underline{\rm x})$ an auxiliary unknown function ${\rm F_A}(\underline{\rm x})$ and to $\mathfrak{M}$ the corresponding equation multiplied by ${\rm F_A}(\underline{\rm x})$; are added to $\underline{\mathfrak{C}}$ the conditions constituted on having ${\rm F_A}(\underline{\rm x})$ the values respectively 1 and 0 in the elements of $\underline{\dot{\mathfrak{X}}}$ and $\underline{\mathfrak{X}} - \underline{\dot{\mathfrak{X}}}$.

6 The approximation of a derivative of the total system with a linear combination of local values of the function to be derived

6.1 The set of rectilinear segments

In relation to $\underline{\mathfrak{X}} \equiv \{\underline{\rm x}_p; p = 1, \hat{\rm P}\}$ said in section 5, are placed

$$\underline{\rm x}_p \equiv \{{\rm x}_{pn}; n = 1, \hat{\rm n}\} \qquad \{a, b\} \subset \{p = 1, \hat{\rm P}\} \qquad {\rm đ}_{ab} \equiv \sqrt{\sum_{n=1}^{\hat{\rm n}} ({\rm x}_{bn} - {\rm x}_{an})^2}$$

$$\vec{\tau}_{ab} \equiv \sum_{n=1}^{\hat{\rm n}} \tau_{abn} \vec{\mathfrak{v}}_n \qquad \tau_{abn} \equiv \frac{{\rm x}_{bn} - {\rm x}_{an}}{{\rm đ}_{ab}}$$

so $đ_{ab}$, $\vec{\tau}_{ab}$, τ_{abn} are respectively the distance between $\underline{x}_a$ and $\underline{x}_b$, the versor of the straight line through these and oriented from $\underline{x}_a$ to $\underline{x}_b$, the n-th cosine director of such line.

It is understood that from $\underline{\mathfrak{X}}$ is deducible a set $\underline{\mathfrak{R}}$ of straight line segments, whose numerosity is maximum compatibly with the

$$\underline{\mathfrak{R}} \equiv \{\underline{R}_r; r = 1, \hat{R}\} \qquad \hat{R} \geq 1 \qquad \{\exists\, \underline{x}_p \in \underline{R}_r; p = 1, \hat{P}\} \qquad Æ\langle \underline{R}_r, r_r, \check{r}_r, \hat{r}_r \,/\!\!/\, \underline{C}, C, \check{C}, \hat{C} \,/\!\!/\, \text{sec. } 3\rangle$$
$$\check{r}_r = R_{r1} = 0 \qquad \exists\, \hat{r}_r = đ_{ab} \qquad \underline{P}_r \equiv \{\underline{x}_{P_{ri}}; i = 1, \hat{i}_r\} = \underline{R}_r \cap \underline{\mathfrak{X}} \qquad \hat{i}_r \geq 3$$
$$\{R_{ri} < R_{r,i+1}; i = 1, \hat{i}_r - 1\} \qquad R_{ri} = đ_{P_{r1}P_{ri}} \tag{44}$$

Are considered the $\{\underline{x}_k; k = 1, \hat{k}\}$ of which $\underline{x}_1 \not\equiv \underline{x}_{\hat{k}}$. The condition for which these $\hat{k}$ points lie on a same straight line is expressed by $\{\vec{\tau}_{1k} = \omega_k \vec{\tau}_{1\hat{k}}; k = 2, \hat{k} - 1\}$, of which $\omega_k \equiv \pm 1$ and

$$\{\vec{\tau}_{1k} = \omega_k \vec{\tau}_{1\hat{k}}; k = 2, \hat{k} - 1\} \equiv$$
$$\Big\{\{\tau_{1kn} = \tau_{1\hat{k}n}; n = 1, \hat{n}\} \,\dot{\vee}\, \{\tau_{1kn} = -\tau_{1\hat{k}n}; n = 1, \hat{n}\}; k = 2, \hat{k} - 1\Big\} \tag{45}$$

on the basis of

$$\{\vec{\tau}_{1k} = \omega_k \vec{\tau}_{1\hat{k}}\} \equiv \{\vec{\tau}_{1k} \cdot \vec{\upsilon}_n = \omega_k \vec{\tau}_{1\hat{k}} \cdot \vec{\upsilon}_n; n = 1, \hat{n}\} \equiv \{\tau_{1kn} = \omega_k \tau_{1\hat{k}n}; n = 1, \hat{n}\} \equiv$$
$$\{\tau_{1kn} = \tau_{1\hat{k}n}; n = 1, \hat{n}\} \,\dot{\vee}\, \{\tau_{1kn} = -\tau_{1\hat{k}n}; n = 1, \hat{n}\}$$

The knowledge of $\{\underline{P}_r; r = 1, \hat{R}\}$ can be achieved with the following algorithm. Are placed $\hat{R} = 0$,

$$\mathcal{A}_{ab\hat{R}} \equiv \Big\{\neg\{\vec{\tau}_{ak} = \omega_k \vec{\tau}_{ab}; k = \alpha_r, \beta_r\}; r = 1, \hat{R}\Big\} \qquad \mathcal{B}_{ab} \equiv \exists \Big\{\vec{\tau}_{ak} = \omega_k \vec{\tau}_{ab} \parallel \underline{x}_k \in \underline{\mathfrak{X}} - \{\underline{x}_a, \underline{x}_b\}\Big\}$$

and are carried out the $\binom{\hat{P}}{2}$ iterations $\{I_{ab}; b = a + 1, \hat{P}; a = 1, \hat{P} - 1\}$. For each I_{ab}, if

$$\Big\{\{\mathcal{A}_{ab\hat{R}} \parallel \hat{R} > 0\} \wedge \mathcal{B}_{ab}\Big\} \,\dot{\bigvee}\, \Big\{\{\hat{R} = 0\} \wedge \mathcal{B}_{ab}\Big\}$$

(this condition can be controlled by means of (45)) $\hat{R}$ is incremented of 1 and are placed $\alpha_{\hat{R}} = a$, $\beta_{\hat{R}} = b$. After these iterations are known the $\{\alpha_r, \beta_r; r = 1, \hat{R}\}$ and are carried out the iterations indicated by $\{r = 1, \hat{R}\}$. For each r, is placed $\hat{i}_r = 0$ and are carried out the iterations indicated by $\{p = 1, \hat{P}\}$. For each p, if

$$\Big\{\vec{\tau}_{\alpha_r p} = \omega_p \vec{\tau}_{\alpha_r \beta_r}\Big\} \bigwedge \Big\{p \notin \{\alpha_r, \beta_r\}\Big\}$$

then $\hat{i}_r$ is incremented of 1 and is placed $P_{r\hat{i}_r} = p$. After these $\hat{P}$ iterations, is modified the subsequent order of the elements of $\{P_{ri}; i = 1, \hat{i}_r\}$ so as to have

$$\exists\, đ_{P_{r1}P_{ri}} = \max\{đ_{P_{ra}P_{rb}}; a = 1, \hat{i}_r; b = 1, \hat{i}_r\} \qquad \{đ_{P_{r1}P_{ri}} < đ_{P_{r1}P_{r,i+1}}; i = 2, \hat{i}_r - 1\}$$

At the end of this algorithm is known the $\{P_{ri}; i = 1, \hat{i}_r; r = 1, \hat{R}\}$ that makes known $\{\underline{P}_r; r = 1, \hat{R}\}$.

6.2 An original algorithm that expresses, by means of a tree graph, the linear combination that approximates a derivative of the total system.

An approximation (43) of a $D_{\mathrm{pd}}(\underline{\tilde{\mathrm{x}}}_{\mathrm{p}})$ (of which (42)) can be obtained using as a logical schema an oriented tree graph implemented by means of

$$\{\text{Æ}\langle \ddot{\mathrm{F}}_{\mathrm{pd},o-1}(\underline{\mathrm{x}}) \,/\!/\, f(\underline{\mathrm{x}}) \,/\!/\, (40)\rangle ; o = 1, \ddot{\mathrm{O}}_{\mathrm{pd}}\}$$

and of the segments $\mathfrak{R}$ of section 6.1.

A graph $\underline{\breve{\mathrm{G}}}$, of which $\underline{\breve{\mathrm{G}}} \equiv \{\underline{\check{\mathrm{N}}}, \underline{\check{\mathrm{A}}}\}$, is constituted by a set of nodes $\underline{\check{\mathrm{N}}}$ of which $\underline{\check{\mathrm{N}}} \equiv \{\check{\mathrm{N}}_{\check{n}}; \check{n} = 1, \hat{\check{n}}\}$, $\hat{\check{n}} \neq \infty$, and by a set of arcs $\underline{\check{\mathrm{A}}}$ of which $\underline{\check{\mathrm{A}}} \equiv \{\check{\mathrm{A}}_{\check{a}}; \check{a} = 1, \hat{\check{a}}\}$, $\hat{\check{a}} \neq \infty$, $\check{\mathrm{A}}_{\check{a}} \equiv (\check{\mathrm{N}}_a, \check{\mathrm{N}}_b)$, $\check{\mathrm{N}}_a \in \underline{\check{\mathrm{N}}}$, $\check{\mathrm{N}}_b \in \underline{\check{\mathrm{N}}}$.

An arc $(\check{\mathrm{N}}_a, \check{\mathrm{N}}_b)$ is directional since it identifies the nodes $\check{\mathrm{N}}_a$ and $\check{\mathrm{N}}_b$ respectively as the origin and destination of an inherent way.

A $\underline{\check{\mathrm{C}}} \equiv \{\check{\mathrm{A}}_{\check{a}_i}; i = 1, \hat{i} - 1\}$, of which $\check{\mathrm{A}}_{\check{a}_i} \equiv (\check{\mathrm{N}}_{\check{n}_i}, \check{\mathrm{N}}_{\check{n}_{i+1}}) \in \underline{\check{\mathrm{A}}}$, defines $\underline{\check{\mathrm{C}}}$ as a path of $\underline{\breve{\mathrm{G}}}$ i.e. as a way that goes from $\check{\mathrm{N}}_{\check{n}_1}$ to $\check{\mathrm{N}}_{\check{n}_{\hat{i}}}$ passing successively for the $\{\check{\mathrm{N}}_{\check{n}_i}; i = 2, \hat{i} - 1\}$.

A $\underline{\breve{\mathrm{G}}}$ is connected if $\{\exists\, \underline{\check{\mathrm{C}}}; \forall \{\check{\mathrm{N}}_{\check{n}_1}, \check{\mathrm{N}}_{\check{n}_{\hat{i}}}\} \subseteq \underline{\check{\mathrm{N}}}\}$. A $\underline{\breve{\mathrm{G}}}$ is oriented or not oriented, respectively if $(\check{\mathrm{N}}_a, \check{\mathrm{N}}_b) \not\equiv (\check{\mathrm{N}}_b, \check{\mathrm{N}}_a)$ or $(\check{\mathrm{N}}_a, \check{\mathrm{N}}_b) \equiv (\check{\mathrm{N}}_b, \check{\mathrm{N}}_a)$. A $\underline{\breve{\mathrm{G}}}$ is a tree if $\hat{\check{a}} = \hat{\check{n}} - 1$ and is connected the graph that is deduced from it by adding to each $(\check{\mathrm{N}}_a, \check{\mathrm{N}}_b)$ a respective $(\check{\mathrm{N}}_b, \check{\mathrm{N}}_a)$. A oriented tree $\underline{\breve{\mathrm{G}}}$ has a root node $\check{\mathrm{N}}_{\mathrm{R}}$, of which $\check{\mathrm{N}}_{\mathrm{R}} \in \underline{\check{\mathrm{N}}}$, that verifies $\{\check{\mathrm{N}} \in \underline{\check{\mathrm{N}}} - \check{\mathrm{N}}_{\mathrm{R}}\} \Rightarrow \exists \left\{\underline{\check{\mathrm{C}}} \parallel \{\check{\mathrm{N}}_{\check{n}_1}, \check{\mathrm{N}}_{\check{n}_{\hat{i}}}\} \equiv \{\check{\mathrm{N}}, \check{\mathrm{N}}_{\mathrm{R}}\}\right\}$ or

$$\{\check{\mathrm{N}} \in \underline{\check{\mathrm{N}}} - \check{\mathrm{N}}_{\mathrm{R}}\} \Rightarrow \exists \left\{\underline{\check{\mathrm{C}}} \parallel \{\check{\mathrm{N}}_{\check{n}_1}, \check{\mathrm{N}}_{\check{n}_{\hat{i}}}\} \equiv \{\check{\mathrm{N}}_{\mathrm{R}}, \check{\mathrm{N}}\}\right\} \tag{46}$$

A $\check{\mathrm{N}}_{\check{n}} \equiv \dot{\mathrm{S}}$ affirms that the object $\dot{\mathrm{S}}$ is associated with the node $\check{\mathrm{N}}_{\check{n}}$. The oriented tree $\underline{\breve{\mathrm{G}}}$, which is used to obtain an approximation (43) of a $D_{\mathrm{pd}}(\underline{\tilde{\mathrm{x}}}_{\mathrm{p}})$, is in particular specified, besides that by $\hat{\check{a}} = \hat{\check{n}} - 1$ and (46), by

$$\underline{\check{\mathrm{N}}} = \{\underline{\check{\mathrm{N}}}_q; q = 0, \ddot{\mathrm{O}}_{\mathrm{pd}}\} \qquad \underline{\check{\mathrm{N}}}_q \equiv \{\check{\mathrm{N}}_{\check{\mathrm{n}}_{q\acute{n}}}; \acute{n} = 1, \hat{\acute{n}}_q\} \qquad \underline{\check{\mathrm{N}}}_0 \equiv \check{\mathrm{N}}_{\check{\mathrm{n}}_{01}} \equiv \check{\mathrm{N}}_{\mathrm{R}} \qquad \neg\exists \{\check{\mathrm{A}}_{\check{a}} \parallel \check{\mathrm{N}}_a \in \underline{\check{\mathrm{N}}}_{\ddot{\mathrm{O}}_{\mathrm{pd}}}\}$$

$$\left\{\neg\exists \left\{\check{\mathrm{A}}_{\check{a}} \parallel \{\check{\mathrm{N}}_a, \check{\mathrm{N}}_b\} \subseteq \underline{\check{\mathrm{N}}}_q\right\}; q = 0, \ddot{\mathrm{O}}_{\mathrm{pd}}\right\} \qquad \left\{\neg\exists \left\{\check{\mathrm{A}}_{\check{a}} \parallel \check{\mathrm{N}}_a \in \underline{\check{\mathrm{N}}}_q, \check{\mathrm{N}}_b \notin \underline{\check{\mathrm{N}}}_{q+1}\right\}; q = 0, \ddot{\mathrm{O}}_{\mathrm{pd}} - 1\right\}$$

$$\underline{\check{\mathrm{A}}} = \{(\check{\mathrm{N}}_{\check{\mathrm{n}}_{q\acute{n}}}, \check{\mathrm{N}}_{\tilde{\check{n}}_{q\acute{n}n\mathfrak{y}}}); \mathfrak{y} = 1, \hat{\mathfrak{y}}_{q\acute{n}n}; n = 1, \hat{\mathrm{n}}; \acute{n} = 1, \hat{\acute{n}}_q; q = 0, \ddot{\mathrm{O}}_{\mathrm{pd}} - 1\}$$

and (with reference to (42) and $\underline{\tilde{\mathrm{x}}}_{\mathrm{p}} \equiv \underline{\mathrm{x}}_{p_{\mathrm{p}}}$) by $\check{\mathrm{N}}_{\check{\mathrm{n}}_{01}} \equiv D_{\mathrm{pd}}(\underline{\mathrm{x}}_{\tilde{p}_{01}})$, $\tilde{p}_{01} = p_{\mathrm{p}}$,

$$\{\check{\mathrm{N}}_{\check{n}_{q\acute{n}}} \equiv \ddot{\mathrm{F}}_{\mathrm{pd},\ddot{\mathrm{o}}_{\mathrm{pd}}-q}(\underline{\mathrm{x}}_{\tilde{p}_{q\acute{n}}}); \acute{n} = 1, \hat{\acute{n}}_q; q = 1, \ddot{\mathrm{O}}_{\mathrm{pd}} - 1\}$$

$$\{\check{\mathrm{N}}_{\check{n}_{\ddot{\mathrm{o}}_{\mathrm{pd}}\acute{n}}} \equiv \mathrm{F}_{m_{\mathrm{pd}}}(\underline{\mathrm{x}}_{\tilde{p}_{\ddot{\mathrm{o}}_{\mathrm{pd}}\acute{n}}}); \acute{n} = 1, \hat{\acute{n}}_{\ddot{\mathrm{O}}_{\mathrm{pd}}}\}$$

of which $\{\tilde{p}_{q\acute{n}} \in \{p = 1, \hat{\mathrm{P}}\}; \acute{n} = 1, \hat{\acute{n}}_q; q = 0, \ddot{\mathrm{O}}_{\mathrm{pd}}\}$ and $\underline{\boldsymbol{\mathfrak{X}}} \equiv \{\underline{\mathrm{x}}_p; p = 1, \hat{\mathrm{P}}\}$, and (with reference to (41) and (44)) by

$$\left\{\left\{\{\ddot{\mathrm{F}}_{\mathrm{pd},\ddot{\mathrm{o}}_{\mathrm{pd}}-q}(\underline{\mathrm{x}}_{\tilde{p}_{q\acute{n}}}) - \ddot{\mathbf{F}}_{\mathrm{pd},\ddot{\mathrm{o}}_{\mathrm{pd}}-q,\tilde{p}_{q\acute{n}}} = 0\} \in \underline{\mathfrak{C}}\right\} \equiv \neg\exists\, (\check{\mathrm{N}}_{\check{n}_{q\acute{n}}}, \check{\mathrm{N}}_b) \in \underline{\check{\mathrm{A}}}; \acute{n} = 1, \hat{\acute{n}}_q; q = 1, \ddot{\mathrm{O}}_{\mathrm{pd}} - 1\right\}$$

$$\left\{\{\check{\mathrm{N}}_{\tilde{\check{n}}_{q\acute{n}n\mathfrak{y}}} \equiv \ddot{\mathrm{F}}_{\mathrm{pd},\ddot{\mathrm{o}}_{\mathrm{pd}}-q-1}(\underline{\mathrm{x}}_{\mathrm{P}_{r_{q\acute{n}n}\mathfrak{y}}}); \mathfrak{y} = 1, \hat{\mathfrak{y}}_{q\acute{n}n}\}, \hat{\mathfrak{y}}_{q\acute{n}n} = \hat{\mathrm{i}}_{r_{q\acute{n}n}}; n = 1, \hat{\mathrm{n}}; \acute{n} = 1, \hat{\acute{n}}_q; q = 0, \ddot{\mathrm{O}}_{\mathrm{pd}} - 1\right\}$$

of which $r_{q\acute{n}n} \in \{r = 1, \hat{\mathrm{R}}\}$, $\underline{\mathfrak{R}} \equiv \{\underline{\mathsf{R}}_r; r = 1, \hat{\mathrm{R}}\}$, $\bigcap_{n=1}^{\hat{\mathrm{n}}} \underline{\mathsf{R}}_{r_{q\acute{n}n}} = \underline{\mathrm{x}}_{\tilde{p}_{q\acute{n}}}$.

The identification of the $\underline{\breve{\mathrm{G}}}$ in question is completed by

$$\Big\{\text{Æ}\Big\langle \ddot{\mathrm{F}}_{\mathrm{pd},\ddot{\mathrm{o}}_{\mathrm{pd}}-q}(\underline{\mathrm{x}}_{\tilde{p}_{q\acute{n}}}), \mathrm{x}_{n_{\mathrm{pd},\ddot{\mathrm{o}}_{\mathrm{pd}}-q}}, \big\{\{\ddot{\mathrm{F}}_{\mathrm{pd},\ddot{\mathrm{o}}_{\mathrm{pd}}-q-1}(\underline{\mathrm{x}}_{\mathrm{P}_{r_{q\acute{n}n}\eta}}); \eta = 1, \hat{\mathrm{i}}_{r_{q\acute{n}n}}\}, \underline{\mathsf{R}}_{r_{q\acute{n}n}}; n = 1, \hat{\mathrm{n}}\big\} /\!\!/ \frac{\partial f(\underline{\mathrm{x}})}{\partial \mathrm{x}_n}, \mathrm{x}_n, \big\{\{\mathsf{f}_{c_{n\bar{\mathrm{B}}}}(\mathrm{C}_{c_{n\bar{\mathrm{B}}}i}); i = 1, \hat{\mathrm{i}}_{c_{n\bar{\mathrm{B}}}}\}, \underline{\mathsf{C}}_{c_{n\bar{\mathrm{B}}}}; n = 1, \hat{\mathrm{n}}\big\} /\!\!/ (40)\Big\rangle; \acute{n} = 1, \hat{\acute{n}}_q; q = 0, \ddot{\mathrm{O}}_{\mathrm{pd}} - 1\Big\}$$

that gives rise to

$$\{\ddot{\mathrm{F}}_{\mathrm{pd},\ddot{\mathrm{o}}_{\mathrm{pd}}-q}(\underline{\mathrm{x}}_{\tilde{p}_{q\acute{n}}}) \approx \sum_{n=1}^{\hat{\mathrm{n}}} \sum_{\eta=1}^{\hat{\mathrm{i}}_{r_{q\acute{n}n}}} \boldsymbol{\Lambda}_{q\acute{n}n\eta} \ddot{\mathrm{F}}_{\mathrm{pd},\ddot{\mathrm{o}}_{\mathrm{pd}}-q-1}(\underline{\mathrm{x}}_{\mathrm{P}_{r_{q\acute{n}n}\eta}}); \acute{n} = 1, \hat{\acute{n}}_q; q = 0, \ddot{\mathrm{O}}_{\mathrm{pd}} - 1\}$$

where each $\boldsymbol{\Lambda}_{q\acute{n}n\eta}$ is known as a specification of the $\mathcal{T}_{\bar{\mathrm{B}}nn}\Lambda_{c_{n\bar{\mathrm{B}}}i}$ of (40), having

$$\big\{\{\check{\mathrm{N}}_{\tilde{n}_{q\acute{n}n\eta}} \equiv \boldsymbol{\Lambda}_{q\acute{n}n\eta}; \eta = 1, \hat{\eta}_{q\acute{n}n}\}, \hat{\eta}_{q\acute{n}n} = \hat{\mathrm{i}}_{r_{q\acute{n}n}}; n = 1, \hat{\mathrm{n}}; \acute{n} = 1, \hat{\acute{n}}_q; q = 0, \ddot{\mathrm{O}}_{\mathrm{pd}} - 1\big\}$$

In what has just been said, is used the set of straight line segments $\underline{\mathfrak{R}}$ instead of a set of curves generally not straight, because (as said in section 3) the maximum absolute value of a derivative defined on a curve is generally less if this is a straight segment, and therefore is generally less also the inherent error (of the type (28)) that influences (as said in section 4) the previous approximation.

Inherently the oriented tree graph $\underline{\breve{\mathrm{G}}}$, is considered the set $\{\underline{\check{\mathrm{C}}}_{\mathsf{k}}; \mathsf{k} = 1, \hat{\mathsf{k}}\}$ of every path $\underline{\check{\mathrm{C}}}_{\mathsf{k}}$ of which $\neg\exists\, \underline{\check{\mathrm{C}}} \supset \underline{\check{\mathrm{C}}}_{\mathsf{k}}$. Each $\underline{\check{\mathrm{C}}}_{\mathsf{k}}$ is a path that goes from $\check{\mathrm{N}}_{\mathrm{R}}$ to a node of $\underline{\check{\mathrm{N}}}_{\ddot{\mathrm{O}}_{\mathrm{pd}}}$ to which is associated an element of $\underline{\mathcal{F}}_{\mathrm{pd}}$ (argument of (43)) or that goes from $\check{\mathrm{N}}_{\mathrm{R}}$ to a node of $\{\underline{\check{\mathrm{N}}}_q; q = 1, \ddot{\mathrm{O}}_{\mathrm{pd}}\}$ to which is associated the known value of an element of $\underline{\mathfrak{C}}$ (of which (41)). Therefore is had

$$\{\underline{\check{\mathrm{C}}}_{\mathsf{k}}; \mathsf{k} = 1, \hat{\mathsf{k}}\} = \big\{\{\underline{\check{\mathrm{C}}}_{\mathsf{k}_a}; a = 1, \hat{a}\}, \{\underline{\check{\mathrm{C}}}_{\mathsf{k}_b}; b = \hat{a} + 1, \hat{\mathsf{k}}\}\big\}$$

whose $\underline{\check{\mathrm{C}}}_{\mathsf{k}_a}$ and $\underline{\check{\mathrm{C}}}_{\mathsf{k}_b}$ are respectively of the two types just now said, and thus the searched approximation (43) of $D_{\mathrm{pd}}(\underline{\widetilde{\mathrm{x}}}_{\mathrm{p}})$ can be the

$$D_{\mathrm{pd}}(\underline{\widetilde{\mathrm{x}}}_{\mathrm{p}}) \approx \tilde{\mathfrak{C}}_{\mathrm{pd}} + \sum_{a=1}^{\hat{a}} \bar{\boldsymbol{\Lambda}}_{\mathsf{k}_a} \mathrm{F}_{m_{\mathrm{pd}}}(\underline{\mathrm{x}}_{\bar{\mathrm{P}}_{\mathrm{pd}a}})$$

of which

$$\tilde{\mathfrak{C}}_{\mathrm{pd}} \equiv \sum_{b=\hat{a}+1}^{\hat{\mathsf{k}}} \bar{\boldsymbol{\Lambda}}_{\mathsf{k}_b} \mathfrak{C}_{\mathsf{k}_b} \qquad \underline{\mathrm{x}}_{\bar{\mathrm{P}}_{\mathrm{pd}a}} \in \underline{\mathfrak{X}} \qquad \mathrm{F}_{m_{\mathrm{pd}}}(\underline{\mathrm{x}}_{\bar{\mathrm{P}}_{\mathrm{pd}a}}) \in \underline{\mathcal{F}}_{\mathrm{pd}}$$

where $\mathfrak{C}_{\mathsf{k}_b}$ is one of the values known in $\underline{\mathfrak{C}}$ and $\bar{\boldsymbol{\Lambda}}_{\mathsf{k}}$ (of which $\mathsf{k} \equiv \mathsf{k}_a \veebar \mathsf{k}_b$) is the product of all factors of type $\boldsymbol{\Lambda}_{q\acute{n}n\eta}$ that are associated with the nodes $\underline{\check{\mathrm{C}}}_{\mathsf{k}}$.

Substituting $\mathrm{F}_{m_{\mathrm{pd}}}(\underline{\mathrm{x}}_{\bar{\mathrm{P}}_{\mathrm{pd}a}})$ with $\sum_{p=1}^{\hat{\mathrm{P}}} \delta_{p\bar{\mathrm{P}}_{\mathrm{pd}a}} \mathrm{F}_{m_{\mathrm{pd}}}(\underline{\mathrm{x}}_p)$ in the previous approximation of $D_{\mathrm{pd}}(\underline{\widetilde{\mathrm{x}}}_{\mathrm{p}})$, is had

$$D_{\mathrm{pd}}(\underline{\widetilde{\mathrm{x}}}_{\mathrm{p}}) \approx \tilde{\mathfrak{C}}_{\mathrm{pd}} + \sum_{p=1}^{\hat{\mathrm{P}}} \tilde{\boldsymbol{\Lambda}}_{\mathrm{pd}p} \mathrm{F}_{m_{\mathrm{pd}}}(\underline{\mathrm{x}}_p) \tag{47}$$

of which $\tilde{\boldsymbol{\Lambda}}_{\mathrm{pd}p} \equiv \sum_{a=1}^{\hat{a}} \bar{\boldsymbol{\Lambda}}_{\mathsf{k}_a} \delta_{p\bar{\mathrm{P}}_{\mathrm{pd}a}}$ and whose $\{\tilde{\boldsymbol{\Lambda}}_{\mathrm{pd}p}; p = 1, \hat{\mathrm{P}}\}$ and $\tilde{\mathfrak{C}}_{\mathrm{pd}}$ can be made known by an algorithm based on

$$Æ\langle \underline{\mathrm{x}}_p, \{\underline{\mathsf{R}}_{\tilde{r}_{pn\mathrm{n}}}; \mathrm{n} = 1, \hat{\mathrm{n}}\} \,/\!\!/\, \underline{\mathrm{x}}, \{\underline{\mathsf{C}}_{c_{\mathrm{n}\tilde{\mathrm{B}}}}; \mathrm{n} = 1, \hat{\mathrm{n}}\} \,/\!\!/\, (40)\rangle$$

which gives rise to

$$\frac{\partial f(\underline{\mathrm{x}}_p)}{\partial \mathrm{x}_n} \approx \sum_{\mathrm{n}=1}^{\hat{\mathrm{n}}} \sum_{i=1}^{\hat{\mathrm{i}}_{\tilde{r}_{pn\mathrm{n}}}} \boldsymbol{\lambda}_{pn\mathrm{n}i} \mathsf{f}_{\tilde{r}_{pn\mathrm{n}}}(\mathsf{r}_{\tilde{r}_{pn\mathrm{n}}i})$$

where each $\boldsymbol{\lambda}_{pn\mathrm{n}i}$ is known and (with reference to (44)) relating to the point $\underline{\mathrm{x}}_{\mathrm{P}_{\tilde{r}_{pn\mathrm{n}}i}}$. Such algorithm is written as follows in a pseudo-language derived from the *Visual Basic*:

$\{\mathcal{B}_p = 0; p = 1, \hat{\mathrm{P}}\}$ $\qquad \mathrm{N} = n_{\mathrm{pd}\ddot{\mathrm{o}}_{\mathrm{pd}}}$

For $\mathrm{n} = 1$ **To** $\hat{\mathrm{n}}$

$\tilde{r} = \tilde{r}_{p_{\mathrm{p}}\mathrm{Nn}}$

  For $i = 1$ **To** $\hat{\mathrm{i}}_{\tilde{r}}$

  $\mathcal{B}_{\mathrm{P}_{\tilde{r}i}} = \boldsymbol{\lambda}_{p_{\mathrm{p}}\mathrm{Nn}i}$

  Next i

Next n

$\tilde{\mathfrak{C}}_{\mathrm{pd}} = 0$ $\qquad \mathrm{IB} \equiv \mathrm{T}$

For $q = 1$ **To** $\ddot{\mathrm{O}}_{\mathrm{pd}} - 1$

$\mathrm{N} = n_{\mathrm{pd},\ddot{\mathrm{o}}_{\mathrm{pd}}-q}$

  If $\mathrm{IB} \equiv \mathrm{T}$ **Then**

  $\mathrm{IB} \equiv \mathrm{F}$

  Call $\mathrm{SubrA}(q, \mathrm{N}, \{\mathcal{A}_p; p = 1, \hat{\mathrm{P}}\}, \{\mathcal{B}_p; p = 1, \hat{\mathrm{P}}\}, \tilde{\mathfrak{C}}_{\mathrm{pd}})$

  Else

  $\mathrm{IB} \equiv \mathrm{T}$

  Call $\mathrm{SubrA}(q, \mathrm{N}, \{\mathcal{B}_p; p = 1, \hat{\mathrm{P}}\}, \{\mathcal{A}_p; p = 1, \hat{\mathrm{P}}\}, \tilde{\mathfrak{C}}_{\mathrm{pd}})$

Next q

If $\mathrm{IB} \equiv \mathrm{T}$ **Then**

Call $\mathrm{SubrB}(\{\mathcal{B}_p; p = 1, \hat{\mathrm{P}}\}, \tilde{\mathfrak{C}}_{\mathrm{pd}}, \{\tilde{\boldsymbol{\Lambda}}_{\mathrm{pd}p}; p = 1, \hat{\mathrm{P}}\})$

Else

Call $\mathrm{SubrB}(\{\mathcal{A}_p; p = 1, \hat{\mathrm{P}}\}, \tilde{\mathfrak{C}}_{\mathrm{pd}}, \{\tilde{\boldsymbol{\Lambda}}_{\mathrm{pd}p}; p = 1, \hat{\mathrm{P}}\})$

End If

  Sub $\mathrm{SubrA}(q, \mathrm{N}, \{\mathcal{A}_p; p = 1, \hat{\mathrm{P}}\}, \{\mathcal{B}_p; p = 1, \hat{\mathrm{P}}\}, \tilde{\mathfrak{C}}_{\mathrm{pd}})$

  $\{\mathcal{A}_p = 0; p = 1, \hat{\mathrm{P}}\}$

For $p = 1$ **To** $\hat{\mathrm{P}}$

  If $\{\ddot{\mathrm{F}}_{\mathrm{pd},\ddot{\mathrm{o}}_{\mathrm{pd}}-q}(\underline{\mathrm{x}}_p) - \ddot{\mathbf{F}}_{\mathrm{pd},\ddot{\mathrm{o}}_{\mathrm{pd}}-q,p} = 0\} \in \underline{\mathfrak{C}}$ **Then**

  $\tilde{\mathfrak{C}}_{\mathrm{pd}} = \tilde{\mathfrak{C}}_{\mathrm{pd}} + \mathcal{B}_p \ddot{\mathbf{F}}_{\mathrm{pd},\ddot{\mathrm{o}}_{\mathrm{pd}}-q,p}$

  Else

    For $\mathrm{n} = 1$ **To** $\hat{\mathrm{n}}$

    $\tilde{r} = \tilde{r}_{p\mathrm{Nn}}$

      For $i = 1$ **To** $\hat{\mathrm{i}}_{\tilde{r}}$

      $\mathcal{A}_{\mathrm{P}_{\tilde{r}i}} = \mathcal{A}_{\mathrm{P}_{\tilde{r}i}} + \mathcal{B}_p \boldsymbol{\lambda}_{p\mathrm{Nn}i}$

      Next i

    Next n

  End If

Next p

End Sub

Sub $\mathrm{SubrB}(\{\mathcal{A}_p; p = 1, \hat{\mathrm{P}}\}, \tilde{\mathfrak{C}}_{\mathrm{pd}}, \{\tilde{\boldsymbol{\Lambda}}_{\mathrm{pd}p}; p = 1, \hat{\mathrm{P}}\})$

For $p = 1$ **To** $\hat{\mathrm{P}}$

  If $\{\mathrm{F}_{m_{\mathrm{pd}}}(\underline{\mathrm{x}}_p) - \tilde{\mathrm{F}}_{m_{\mathrm{pd}}p} = 0\} \in \underline{\mathfrak{C}}$ **Then**

  $\tilde{\mathfrak{C}}_{\mathrm{pd}} = \tilde{\mathfrak{C}}_{\mathrm{pd}} + \mathcal{A}_p \tilde{\mathrm{F}}_{m_{\mathrm{pd}}p}$

  $\tilde{\boldsymbol{\Lambda}}_{\mathrm{pd}p} = 0$

  Else

  $\tilde{\boldsymbol{\Lambda}}_{\mathrm{pd}p} = \mathcal{A}_p$

  End If

Next p

End Sub

7 Conclusion

The definitions, procedures and results presented in this paper have been used to implement a computer program to which has been given the name PEEI (as acronym for "Programma agli Elementi di Estensione Infinitesima") and which is aimed at numerically solve any differential analytical model.

More precisely, this program calculates (if is not impossible) a numerical solution of every system of partial differential equations, with number of equations not lower than that of its functions unknowns, and subjected to any additional condition like those initial or boundary.

The program PEEI is freeware and available in *http://www.giacomo.lorenzoni.name/peei/* together with numerous application examples that confirm its reliability through the comparison of its solutions with those exact.

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