

Plane Poiseuille Flow

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http://www.giacomo.lorenzoni.name/PEEI_4.0.0.1/Plane_Poiseuille_flow/

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Plane Poiseuille flow

This text is integrating part of the homonymous link in [PEEI: a computer program for the numerical solution of systems of partial differential equations](#).

System of measurement: International System of Units.

Coordinate system: Cartesian

Coordinates: \underline{x} of which $\underline{x} = \{x_i; i=1,3\}$ $[x_i] = [\text{length}]$ $\mathcal{R}(\underline{x}_i) = (-\infty, \infty)$

Coordinate versors: $\{\mathbf{v}_i; i=1,3\}$

Unknown functions: $\{w_1, w_2, w_3, P\}$ of which $\mathbf{w} = \sum_{i=1,3} (w_i \cdot \mathbf{v}_i)$, $[w_i] = [\text{speed}]$, \mathbf{w} the velocity vector, $[P] = [\text{pressure}]$.

Differential analytical model:

$$\sum_{i=1,3} (\partial w_i / \partial x_i) = 0 \quad (1)$$

$$\{\rho \cdot (\sum_{j=1,3} (w_j \cdot (\partial w_i / \partial x_j)) - F_i) + \partial P / \partial x_i - \mu \cdot \sum_{j=1,3} (\sum_{h=1,3} (\sum_{k=1,3} (\delta_{jikh} \cdot (\partial^2 w_k / \partial x_h \partial x_j)))) = 0; i=1,3\} \quad (2)$$

of which: $\rho = 998.2071$ $[\rho] = [\text{density}]$, $\mathbf{F} = \sum_{i=1,3} (F_i \cdot \mathbf{v}_i)$, $[F_i] = [\text{force/mass}]$, \mathbf{F} the body force vector per unit mass, $\mu = 0.001003$ $[\mu] = [\text{dynamic viscosity}]$, $\delta_{ijkh} = \delta_{ik} \cdot \delta_{jh} + \delta_{jk} \cdot \delta_{ih} - (2/3) \cdot \delta_{hk} \cdot \delta_{ij}$, $\{\delta_{ij} = 0; \forall i \neq j\}$ $\{\delta_{ij} = 1; \forall i = j\}$. The (1) is the continuity equation for incompressible fluids, the (2) are the stationary incompressible Navier-Stokes equations for constant viscosity.

Definition set: $\{\underline{x} / 0 \leq x_1 \leq L_1; 0 \leq x_2 \leq L_2; 0 \leq x_3 \leq L_3\}$ $L_1 = 1$ $L_2 = 1000$ $L_3 = 0.1$.

Conditions:

$$F_1 = F_2 = 0 \quad F_3 = -9.80665 \quad w_1(\underline{x}) = w_3(\underline{x}) = w_2(x_1, x_2, 0) = w_2(x_1, x_2, L_3) = \partial w_2(\underline{x}) / \partial x_1 = 0 \quad \partial P(\underline{x}) / \partial x_2 = K = -1 \quad P(0, 0, L_3) = 1000000 \quad (3)$$

Related files: [mad.txt](#)

Exact solution:

From (1) (2) and (3) follows $\partial w_2(\underline{x}) / \partial x_2 = \partial P(\underline{x}) / \partial x_1 = 0$, $\mu \cdot (\partial^2 w_2(\underline{x}) / \partial x_3^2) = K$, $\partial P(\underline{x}) / \partial x_3 = \rho \cdot F_3$, and then

$$dP(\underline{x}) = \sum_{i=1,3} ((\partial P(\underline{x}) / \partial x_i) \cdot dx_i) = K \cdot dx_2 + \rho \cdot F_3 \cdot dx_3 \quad (4)$$

Are placed

$$A \leq c \leq B \quad \underline{x}(c) = \{x_i(c); i=1,3\} \quad \underline{x}_A = \{0, 0, L_3\} = \{x_{Ai}; i=1,3\} = \underline{x}(A) \quad \underline{x}_B = \{x_{Bi}; i=1,3\} = \underline{x}(B)$$

These and (4) imply $dP(\underline{x}(c)) = K \cdot dx_2(c) + \rho \cdot F_3 \cdot dx_3(c) = (K \cdot x_2'(c) + \rho \cdot F_3 \cdot x_3'(c)) \cdot dc$ and then

$$P(\underline{x}_B)=P(\underline{x}_A)+\int_{A,B}((K\cdot x_2'(c)+\rho\cdot F_3\cdot x_3'(c))\cdot dc) \quad (5)$$

Are placed

$$\begin{aligned} \int_{A,B}(f(c)\cdot dc)=\int_{A,N}(f(c)\cdot dc)+\int_{N,Q}(f(c)\cdot dc)+\int_{Q,B}(f(c)\cdot dc) \quad \underline{x}(N)\equiv\{0,0,x_{B3}\} \quad \underline{x}(Q)\equiv\{x_{B1},0,x_{B3}\} \\ \{x_1'(c)=x_2'(c)=0, x_3'(c)=-1; \forall c\in[A,N]\} \quad \{x_2'(c)=x_3'(c)=0, x_1'(c)=1; \forall c\in[N,Q]\} \\ \{x_1'(c)=x_3'(c)=0, x_2'(c)=1; \forall c\in[Q,B]\} \end{aligned} \quad (6)$$

that imply $\int_{A,B}((K\cdot x_2'(c)+\rho\cdot F_3\cdot x_3'(c))\cdot dc)=K\cdot x_{B2}+\rho\cdot F_3\cdot(x_{B3}-L_3)$. From this and (5) follows

$$P(\underline{x})=P(\underline{x}_A)+K\cdot x_2+\rho\cdot F_3\cdot(x_3-L_3) \quad (7)$$

The $\mu\cdot(\partial^2 w_2(\underline{x})/\partial x_3^2)-K=0$ implies $\mu\cdot(\partial w_2(\underline{x})/\partial x_3)-K\cdot x_3+R(x_1,x_2)=0$, and then $\mu\cdot w_2(\underline{x})-K\cdot x_3^2/2+R(x_1,x_2)\cdot x_3=0$. This and $w_2(x_1,x_2,L_3)=0$ imply $R(x_1,x_2)=K\cdot L_3/2$. Hence

$$w_2(\underline{x})=K\cdot x_3\cdot(x_3-L_3)/(2\cdot\mu) \quad (8)$$

Note: In the following diagrams, the symbols \square (empty square) and \bullet (full circle) are respectively inherent to the solution expressed by (7) and (8), and the solution calculated by PEEI.

Case 1: [points-1.txt](#), [mem-1.bin](#), [cond-1.txt](#), [sol-1.txt](#), [plot-1-1.jpg](#), [plot-1-2.jpg](#)

Case 2: [points-2.txt](#), [mem-2.bin](#), [cond-2.txt](#), [sol-2.txt](#), [plot-2-1.jpg](#), [plot-2-2.jpg](#)

Case 3: [points-3.txt](#), [mem-3.bin](#), [cond-3.txt](#), [sol-3.txt](#), [plot-3-1.jpg](#), [plot-3-2.jpg](#)

Case 4: [points-4.txt](#), [mem-4.bin](#), [cond-4.txt](#), [sol-4.txt](#), [plot-4-1.jpg](#), [plot-4-2.jpg](#)

Case 5: [points-5.txt](#), [mem-5.bin](#), [cond-5.txt](#), [sol-5.txt](#), [plot-5-1.jpg](#), [plot-5-2.jpg](#)

Case 6: [points-6.txt](#), [mem-6.bin](#), [cond-6.txt](#), [sol-6.txt](#), [plot-6-1.jpg](#), [plot-6-2.jpg](#)

Case 7: [points-7.txt](#), [mem-7.bin](#), [cond-7.txt](#), [sol-7.txt](#), [plot-7-1.jpg](#), [plot-7-2.jpg](#)

Case 8: [points-8.txt](#), [mem-8.bin](#), [cond-8.txt](#), [sol-8.txt](#), [plot-8-1.jpg](#), [plot-8-2.jpg](#)

Case 9: [points-9.txt](#), [mem-9.bin](#), [cond-9.txt](#), [sol-9.txt](#), [plot-9-1.jpg](#), [plot-9-2.jpg](#)

Case 10: [points-10.txt](#), [mem-10.bin](#), [cond-10.txt](#), [sol-10.txt](#), [plot-10-1.jpg](#), [plot-10-2.jpg](#)

Case 11: [points-11.txt](#), [mem-11.bin](#), [cond-11.txt](#), [sol-11.txt](#), [plot-11-1.jpg](#), [plot-11-2.jpg](#)

Case 12: [points-12.txt](#), [mem-12.bin](#), [cond-12.txt](#), [sol-12.txt](#), [plot-12-1.jpg](#), [plot-12-2.jpg](#)

Case 13: [points-13.txt](#), [mem-13.bin](#), [cond-13.txt](#), [sol-13.txt](#), [plot-13-1.jpg](#), [plot-13-2.jpg](#)

Case 14: [points-14.txt](#), [mem-14.bin](#), [cond-14.txt](#), [sol-14.txt](#), [plot-14-1.jpg](#), [plot-14-2.jpg](#)

Case 15: [points-15.txt](#), [mem-15.bin](#), [cond-15.txt](#), [sol-15.txt](#), [plot-15-1.jpg](#), [plot-15-2.jpg](#)

Case 16: [points-16.txt](#), [mem-16.bin](#), [cond-16.txt](#), [sol-16.txt](#), [plot-16-1.jpg](#), [plot-16-2.jpg](#)

Case 17: [points-17.txt](#), [mem-17.bin](#), [cond-17.txt](#), [sol-17.txt](#), [plot-17-1.jpg](#), [plot-17-2.jpg](#)