

Laminar Taylor-Couette Flow

14/11/2008

Giacomo Lorenzoni

http://www.giacomo.lorenzoni.name/PEEI_4.0.0.1/Laminar_Taylor-Couette_flow/

http://www.giacomo.lorenzoni.name/PEEI_4.0.0.1/PEEIapplDown.aspx?var=12

Laminar Taylor-Couette flow

This text is integrating part of the homonymous link in [PEEI: a computer program for the numerical solution of systems of partial differential equations](#).

System of measurement: International System of Units

Coordinate system: Cylindrical

Coordinates: \underline{c} of which $\underline{c} \equiv \{c_i; i=1,3\}$ $\mathbb{R}\langle c_1 \rangle \equiv [0, \infty)$ $\mathbb{R}\langle c_2 \rangle \equiv [0, 2\pi)$ $\mathbb{R}\langle c_3 \rangle \equiv (-\infty, \infty)$

Coordinate versors: $\{\mathbf{\kappa}_i; i=1,3\}$

Unknown functions: $\{W_1, W_2, W_3, P\}$ of which $\mathbf{W} \equiv \sum_{i=1,3} (W_i \cdot \mathbf{\kappa}_i)$, $[W_i] \equiv [\text{speed}]$, \mathbf{W} the velocity vector, $[P] \equiv [\text{pressure}]$.

Deduction of the differential analytical model:

The $c_1 = c_1(\underline{x}) \equiv (x_1^2 + x_2^2)^{0.5}$ $c_2 = c_2(\underline{x}) \equiv \arcsin(x_2 / (x_1^2 + x_2^2)^{0.5}) = \arctan\langle x_2 / x_1 \rangle + K$ $c_3 = c_3(\underline{x}) \equiv x_3$ of which $\underline{x} \equiv \{x_i; i=1,3\}$ $\mathbb{R}\langle x_i \rangle \equiv (-\infty, \infty)$, imply the $x_1 = x_1(\underline{c}) \equiv c_1 \cdot \cos\langle c_2 \rangle$ $x_2 = x_2(\underline{c}) \equiv c_1 \cdot \sin\langle c_2 \rangle$ $x_3 = x_3(\underline{c}) \equiv c_3$, where \underline{x} are the coordinates of a Cartesian coordinate system that have the versors $\{\mathbf{v}_i; i=1,3\}$, and K is a constant determined by the trigonometric quadrant of $\{x_1, x_2\}$. Hence

$$\begin{aligned} \partial c_1 / \partial x_1 &= \cos\langle c_2 \rangle = x_1 / c_1 \quad \partial c_1 / \partial x_2 = \sin\langle c_2 \rangle = x_2 / c_1 \quad \partial c_2 / \partial x_1 = -\sin\langle c_2 \rangle / c_1 = -x_2 / c_1^2 \\ \partial c_2 / \partial x_2 &= \cos\langle c_2 \rangle / c_1 = x_1 / c_1^2 \quad \partial c_1 / \partial x_3 = \partial c_2 / \partial x_3 = \partial c_3 / \partial x_1 = \partial c_3 / \partial x_2 = 0 \quad \partial c_3 / \partial x_3 = 1 \end{aligned}$$

$$\partial_{hi} \equiv \partial c_h / \partial x_i = \hat{o}_{h1i3} \cdot \hat{o}_{h2i3} \cdot \hat{o}_{h3i1} \cdot \hat{o}_{h3i2} \cdot ((\check{o}_{h1i1} \cdot x_1 + \check{o}_{h1i2} \cdot x_2) / c_1 + (\check{o}_{h2i2} \cdot x_1 - \check{o}_{h2i1} \cdot x_2) / c_1^2 + \check{o}_{h3i3})$$

$$\begin{aligned} \partial_{hij}^2 &\equiv \partial^2 c_h / \partial x_i \partial x_j = \hat{o}_{h1i3} \cdot \hat{o}_{h1j3} \cdot \hat{o}_{h2i3} \cdot \hat{o}_{h2j3} \cdot \hat{o}_{h3i1} \cdot \hat{o}_{h3j1} \cdot \hat{o}_{h3i2} \cdot \hat{o}_{h3j2} \cdot \\ &(\check{o}_{j1} \cdot (\check{o}_{h1i1} / c_1 - (\check{o}_{h1i1} \cdot x_1 + \check{o}_{h1i2} \cdot x_2) \cdot x_1 / c_1^3 + \check{o}_{h2i2} / c_1^2 - 2 \cdot (\check{o}_{h2i2} \cdot x_1 - \check{o}_{h2i1} \cdot x_2) \cdot x_1 / c_1^4) + \\ &\check{o}_{j2} \cdot (\check{o}_{h1i2} / c_1 - (\check{o}_{h1i1} \cdot x_1 + \check{o}_{h1i2} \cdot x_2) \cdot x_2 / c_1^3 - \check{o}_{h2i1} / c_1^2 - 2 \cdot (\check{o}_{h2i2} \cdot x_1 - \check{o}_{h2i1} \cdot x_2) \cdot x_2 / c_1^4)) \end{aligned}$$

where $\check{o}_{ijk} = 1 - \hat{o}_{ijk}$, $\hat{o}_{ijk} = 0$ if $i=j$ $h=k$ and otherwise $\hat{o}_{ijk} = 1$.

The $\mathbf{\kappa}_i \equiv \sum_{j=1,3} (\mathbf{\kappa}_i \cdot \mathbf{v}_j)$ implies $\mathbf{\kappa}_i \cdot \mathbf{v}_j = \sum_{i=1,3} (\mathbf{\kappa}_i \cdot \mathbf{v}_i \cdot \mathbf{v}_j)$. This and $\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$ (where $\{\delta_{ij} = 0; \forall i \neq j\}$ $\{\delta_{ij} = 1; \forall i = j\}$), imply $\mathbf{\kappa}_i \cdot \mathbf{v}_j = \kappa_{ij}$ and then $\kappa_{ij} = \cos\langle \alpha_{ij} \rangle$ where α_{ij} is the angle between $\mathbf{\kappa}_i$ and \mathbf{v}_j . Hence

$$\kappa_{11} = \kappa_{22} = x_1 / c_1 \quad \kappa_{33} = 1 \quad \kappa_{12} = -\kappa_{21} = x_2 / c_1 \quad \kappa_{13} = \kappa_{23} = \kappa_{31} = \kappa_{32} = 0$$

$$\kappa_{ij} = \hat{o}_{i1j3} \cdot \hat{o}_{i2j3} \cdot \hat{o}_{i3j1} \cdot \hat{o}_{i3j2} \cdot ((\check{o}_{i1j1} + \check{o}_{i2j2}) \cdot x_1 / c_1 + (\check{o}_{i1j2} - \check{o}_{i2j1}) \cdot x_2 / c_1 + \check{o}_{i3j3})$$

$$\begin{aligned} \partial_{ijh} &\equiv \partial \kappa_{ij} / \partial x_h = \hat{o}_{i1j3} \cdot \hat{o}_{i2j3} \cdot \hat{o}_{i3j1} \cdot \hat{o}_{i3j2} \cdot (\check{o}_{h1} \cdot ((\check{o}_{i1j1} + \check{o}_{i2j2}) / c_1 - ((\check{o}_{i1j1} + \check{o}_{i2j2}) \cdot x_1 + (\check{o}_{i1j2} - \check{o}_{i2j1}) \cdot x_2) \cdot x_1 / c_1^3) + \\ &\check{o}_{h2} \cdot ((\check{o}_{i1j2} - \check{o}_{i2j1}) / c_1 - ((\check{o}_{i1j1} + \check{o}_{i2j2}) \cdot x_1 + (\check{o}_{i1j2} - \check{o}_{i2j1}) \cdot x_2) \cdot x_2 / c_1^3)) \end{aligned}$$

$$\begin{aligned} \partial^2_{ijk} \equiv \partial^2_{kij} / \partial x_h \partial x_k = & -\hat{o}_{i1j3} \cdot \hat{o}_{i2j3} \cdot \hat{o}_{i3j1} \cdot \hat{o}_{i3j2} \cdot (\bar{\delta}_{k1} \cdot (\bar{\delta}_{h1} \cdot (2 \cdot (\bar{o}_{i1j1} + \bar{o}_{i2j2}) \cdot x_1 / C_1^3 + \\ & ((\bar{o}_{i1j1} + \bar{o}_{i2j2}) \cdot x_1 + (\bar{o}_{i1j2} - \bar{o}_{i2j1}) \cdot x_2) \cdot (C_1^{-3} - 3 \cdot x_1^2 / C_1^5)) + \\ & \bar{\delta}_{h2} \cdot ((\bar{o}_{i1j2} - \bar{o}_{i2j1}) \cdot x_1 / C_1^3 + (\bar{o}_{i1j1} + \bar{o}_{i2j2}) \cdot x_2 / C_1^3 - 3 \cdot ((\bar{o}_{i1j1} + \bar{o}_{i2j2}) \cdot x_1 + (\bar{o}_{i1j2} - \bar{o}_{i2j1}) \cdot x_2) \cdot x_1 \cdot x_2 / C_1^5)) + \\ & \bar{\delta}_{k2} \cdot (\bar{\delta}_{h1} \cdot ((\bar{o}_{i1j1} + \bar{o}_{i2j2}) \cdot x_2 / C_1^3 + (\bar{o}_{i1j2} - \bar{o}_{i2j1}) \cdot x_1 / C_1^3 - 3 \cdot ((\bar{o}_{i1j1} + \bar{o}_{i2j2}) \cdot x_1 + (\bar{o}_{i1j2} - \bar{o}_{i2j1}) \cdot x_2) \cdot x_1 \cdot x_2 / C_1^5)) + \\ & \bar{\delta}_{h2} \cdot (2 \cdot (\bar{o}_{i1j2} - \bar{o}_{i2j1}) \cdot x_2 / C_1^3 + ((\bar{o}_{i1j1} + \bar{o}_{i2j2}) \cdot x_1 + (\bar{o}_{i1j2} - \bar{o}_{i2j1}) \cdot x_2) \cdot (C_1^{-3} - 3 \cdot x_2^2 / C_1^5)))) \end{aligned}$$

The $\mathbf{v}_i \cdot \mathbf{v}_j = \bar{\delta}_{ij}$ $\mathbf{k}_i \cdot \mathbf{v}_j = \kappa_{ij}$ $\sum_{j=1,3} (\mathbf{v}_j \cdot \mathbf{k}_j) = \sum_{j=1,3} (\mathbf{v}_j \cdot \mathbf{v}_j)$, imply

$$\mathbf{v}_i = \sum_{j=1,3} (\mathbf{v}_j \cdot \mathbf{k}_{ji}) \quad (1)$$

From $F(\underline{x}) \equiv F(\underline{c}(\underline{x}))$ where $\underline{c}(\underline{x}) \equiv \{c_i(\underline{x}); i=1,3\}$, follows

$$\partial F(\underline{x}) / \partial x_i = \partial F(\underline{c}(\underline{x})) / \partial x_i = \sum_{i=1,3} ((\partial F(\underline{c}(\underline{x})) / \partial c_i) \cdot (\partial c_i(\underline{x}) / \partial x_i)) \equiv \sum_{i=1,3} ((\partial F / \partial c_i) \cdot \partial_{ii}) \quad (2)$$

$$\begin{aligned} \partial^2 F(\underline{x}) / \partial x_i \partial x_j = & \partial^2 F(\underline{c}(\underline{x})) / \partial x_i \partial x_j = \partial (\partial F(\underline{c}(\underline{x})) / \partial x_i) / \partial x_j = \sum_{i=1,3} (\partial ((\partial F(\underline{c}(\underline{x})) / \partial c_i) \cdot (\partial c_i(\underline{x}) / \partial x_i)) / \partial x_j) = \\ & \sum_{i=1,3} ((\partial F(\underline{c}(\underline{x})) / \partial c_i) \cdot (\partial^2 c_i(\underline{x}) / \partial x_i \partial x_j) + (\partial c_i(\underline{x}) / \partial x_i) \cdot \sum_{j=1,3} ((\partial^2 F(\underline{c}(\underline{x})) / \partial c_i \partial c_j) \cdot (\partial c_j(\underline{x}) / \partial x_j))) = \\ & \sum_{i=1,3} ((\partial F / \partial c_i) \cdot \partial^2_{ii} + \sum_{j=1,3} (\partial_{ii} \cdot \partial_{jj} \cdot (\partial^2 F / \partial c_i \partial c_j))) \end{aligned} \quad (3)$$

The continuity equation for incompressible fluids and the stationary incompressible Navier-Stokes equations for constant viscosity, in Cartesian coordinates are respectively

$$\sum_{i=1,3} (\partial w_i / \partial x_i) = 0 \quad (4)$$

$$\{\rho \cdot (\sum_{j=1,3} (\mathbf{w}_j \cdot (\partial \mathbf{w}_i / \partial x_j)) - F_i) + \partial P / \partial x_i - \mu \cdot \sum_{j=1,3} (\sum_{h=1,3} (\sum_{k=1,3} (\delta_{jikh} \cdot (\partial^2 \mathbf{w}_k / \partial x_h \partial x_j)))) = 0; i=1,3\} \quad (5)$$

of which: $\mathbf{w} = \sum_{i=1,3} (w_i \cdot \mathbf{v}_i)$, $[w_i] = [\text{speed}]$, $[\rho] = [\text{density}]$, $\mathbf{F} = \sum_{i=1,3} (F_i \cdot \mathbf{v}_i)$, $[F_i] = [\text{force/mass}]$, \mathbf{F} the body force vector per unit mass, $[\mu] = [\text{dynamic viscosity}]$, $\delta_{ijkh} \equiv \bar{\delta}_{ik} \cdot \bar{\delta}_{jh} + \bar{\delta}_{jk} \cdot \bar{\delta}_{ih} - (2/3) \cdot \bar{\delta}_{hk} \cdot \bar{\delta}_{ij}$.

From $\sum_{i=1,3} (\mathbf{w}_i \cdot \mathbf{v}_i) = \sum_{i=1,3} (\mathbf{w}_i \cdot \mathbf{k}_i)$ and (1) follows $\mathbf{w}_i = \sum_{j=1,3} (\mathbf{w}_j \cdot \mathbf{k}_{ji})$. This, (4) (5) (2) and (3), imply (6) and (7).

Differential analytical model:

$$\sum_{i=1,3} (\sum_{j=1,3} (\partial_{jii} \cdot \mathbf{w}_j) + \sum_{h=1,3} (\kappa_{ji} \cdot \partial_{hi} \cdot (\partial \mathbf{w}_j / \partial c_h))) = 0 \quad (6)$$

$$\begin{aligned} & \{\sum_{j=1,3} (\partial_{jii} \cdot (\partial P / \partial c_j)) + \\ & \sum_{h=1,3} (\sum_{k=1,3} (\rho \cdot \kappa_{hj} \cdot \partial_{kij} \cdot \mathbf{w}_h \cdot \mathbf{w}_k + \\ & \sum_{m=1,3} (\rho \cdot \kappa_{hj} \cdot \kappa_{ki} \cdot \partial_{mj} \cdot \mathbf{w}_h \cdot (\partial \mathbf{w}_k / \partial c_m) - \mu \cdot \delta_{jikh} \cdot \partial^2_{mkhj} \cdot \mathbf{w}_m - \\ & \mu \cdot \delta_{jikh} \cdot \sum_{n=1,3} ((\partial_{nj} \cdot \partial_{mkn} + \partial_{nh} \cdot \partial_{mkj} + \kappa_{mk} \cdot \partial^2_{nhj}) \cdot (\partial \mathbf{w}_m / \partial c_n)) + \\ & \sum_{p=1,3} (\kappa_{mk} \cdot \partial_{nh} \cdot \partial_{pj} \cdot (\partial^2 \mathbf{w}_m / \partial c_n \partial c_p)))))) - \rho \cdot F_i = 0; i=1,3\} \end{aligned} \quad (7)$$

of which $\rho = 998.2071$ $\mu = 0.001003$. The (6) and (7) are respectively, in cylindrical coordinates, the continuity equation for incompressible fluids and the stationary incompressible Navier-Stokes equations for constant viscosity.

Known functions: $\{F_i; i=1,156\}$ of which $\{F_i = F_i = 0; i=1,3\}$ $F_{A(h,i)} = \partial_{hi}$ $F_{B(h,i,j)} = \partial^2_{hij}$ $F_{C(i,j)} = \kappa_{ij}$ $F_{D(i,j,h)} = \partial_{ijh}$ $F_{E(i,j,h,k)} = \partial^2_{ijkh}$ $A_{hi} = 3 + h + 3 \cdot (i-1)$ $B_{hij} = A_{33} + h + 3 \cdot (i-1) + 9 \cdot (j-1)$ $C_{ij} = B_{333} + i + 3 \cdot (j-1)$ $D_{ijh} = C_{33} + i + 3 \cdot (j-1) + 9 \cdot (h-1)$ $E_{ijkh} = D_{333} + i + 3 \cdot (j-1) + 9 \cdot (h-1) + 27 \cdot (k-1)$.

Definition set: $\{\underline{c} / R_1 \leq c_1 \leq R_2; 0 \leq c_2 < 2 \cdot \pi; 0 \leq c_3 \leq L_3\}$ $R_1 = 1$ $R_2 = 4$ $L_3 = 1000$.

Conditions: $W_1(\underline{C})=W_3(\underline{C})=\partial W_2/\partial C_2=\partial P(\underline{C})/\partial C_2=0$ $W_2(R_1, C_2, C_3)=W_1=1$ $W_2(R_2, C_2, C_3)=W_E=0.9$
 $P(R_1, C_2, C_3)=P_1=1000000$ (8)

Related files: [mad.txt](#)

Exact solution:

From [here](#) follows

$$W_2=W_2(C_1)\equiv(W_1\cdot(R_2/C_1-C_1/R_2)+W_E\cdot(C_1/R_1-R_1/C_1))/(R_2/R_1-R_1/R_2) \quad (9)$$

The equilibrium between pressure and centrifugal forces (and the $\partial P(\underline{C})/\partial C_2=\partial P(\underline{C})/\partial C_3=0$ and (9)) imply $dP=(W_2^2/C_1)\cdot dC_1$. From this and (9) follows

$$P=p\cdot(2^{-1}\cdot C^2\cdot C_1^{-2}+2\cdot B\cdot C\cdot\ln(C_1)-2^{-1}\cdot B^2\cdot C_1^{-2})\cdot A^{-2}+D \quad (10)$$

where $A\equiv(R_2/R_1-R_1/R_2)$ $B\equiv(W_1\cdot R_2-W_E\cdot R_1)$ $C\equiv(W_E/R_1-W_1/R_2)$ $D\equiv P_1-p\cdot(2^{-1}\cdot C^2\cdot R_1^{-2}+2\cdot B\cdot C\cdot\ln(R_1)-2^{-1}\cdot B^2\cdot R_1^{-2})\cdot A^{-2}$

Note: In the following diagrams, the continuous line and the symbol \bullet (full circle) are respectively inherent to the (9) and (10), and the solution calculated by PEEI.

Case 1: [points-1.txt](#), points-1.bin, [cond-1.txt](#), [sol-1.txt](#), [plot-1-1.jpg](#), [plot-1-2.jpg](#)

Case 2: [points-2.txt](#), points-2.bin, [cond-2.txt](#), [sol-2.txt](#), [plot-2-1.jpg](#), [plot-2-2.jpg](#)

Case 3: [points-3.txt](#), points-3.bin, [cond-3.txt](#), [sol-3.txt](#), [plot-3-1.jpg](#), [plot-3-2.jpg](#)

Case 4: [points-4.txt](#), points-4.bin, [cond-4.txt](#), [sol-4.txt](#), [plot-4-1.jpg](#), [plot-4-2.jpg](#)

Case 5: [points-5.txt](#), points-5.bin, [cond-5.txt](#), [sol-5.txt](#), [plot-5-1.jpg](#), [plot-5-2.jpg](#)

Case 6: [points-6.txt](#), points-6.bin, [cond-6.txt](#), [sol-6.txt](#), [plot-6-1.jpg](#), [plot-6-2.jpg](#)

Case 7: [points-7.txt](#), points-7.bin, [cond-7.txt](#), [sol-7.txt](#), [plot-7-1.jpg](#), [plot-7-2.jpg](#)

Case 8: [points-8.txt](#), points-8.bin, [cond-8.txt](#), [sol-8.txt](#), [plot-8-1.jpg](#), [plot-8-2.jpg](#)

Case 9: [points-9.txt](#), points-9.bin, [cond-9.txt](#), [sol-9.txt](#), [plot-9-1.jpg](#), [plot-9-2.jpg](#)

Case 10: [points-10.txt](#), points-10.bin, [cond-10.txt](#), [sol-10.txt](#), [plot-10-1.jpg](#), [plot-10-2.jpg](#)

Case 11: [points-11.txt](#), points-11.bin, [cond-11.txt](#), [sol-11.txt](#), [plot-11-1.jpg](#), [plot-11-2.jpg](#)

Case 12: [points-12.txt](#), points-12.bin, [cond-12.txt](#), [sol-12.txt](#), [plot-12-1.jpg](#), [plot-12-2.jpg](#)

Case 13: [points-13.txt](#), points-13.bin, [cond-13.txt](#), [sol-13.txt](#), [plot-13-1.jpg](#), [plot-13-2.jpg](#)

Case 14: [points-14.txt](#), points-14.bin, [cond-14.txt](#), [sol-14.txt](#), [plot-14-1.jpg](#), [plot-14-2.jpg](#)

Case 15: [points-15.txt](#), points-15.bin, [cond-15.txt](#), [sol-15.txt](#), [plot-15-1.jpg](#), [plot-15-2.jpg](#)

Case 16: [points-16.txt](#), points-16.bin, [cond-16.txt](#), [sol-16.txt](#), [plot-16-1.jpg](#), [plot-16-2.jpg](#)

Case 17: [points-17.txt](#), points-17.bin, [cond-17.txt](#), [sol-17.txt](#), [plot-17-1.jpg](#), [plot-17-2.jpg](#)

Case 18: [points-18.txt](#), points-18.bin, [cond-18.txt](#), [sol-18.txt](#), [plot-18-1.jpg](#), [plot-18-2.jpg](#)
Case 19: [points-19.txt](#), points-19.bin, [cond-19.txt](#), [sol-19.txt](#), [plot-19-1.jpg](#), [plot-19-2.jpg](#)
Case 20: [points-20.txt](#), points-20.bin, [cond-20.txt](#), [sol-20.txt](#), [plot-20-1.jpg](#), [plot-20-2.jpg](#)
Case 21: [points-25.txt](#), points-25.bin, [cond-25.txt](#), [sol-25.txt](#), [plot-25-1.jpg](#), [plot-25-2.jpg](#)
Case 22: [points-30.txt](#), points-30.bin, [cond-30.txt](#), [sol-30.txt](#), [plot-30-1.jpg](#), [plot-30-2.jpg](#)
Case 23: [points-35.txt](#), points-35.bin, [cond-35.txt](#), [sol-35.txt](#), [plot-35-1.jpg](#), [plot-35-2.jpg](#)
Case 24: [points-40.txt](#), points-40.bin, [cond-40.txt](#), [sol-40.txt](#), [plot-40-1.jpg](#), [plot-40-2.jpg](#)
Case 25: [points-45.txt](#), points-45.bin, [cond-45.txt](#), [sol-45.txt](#), [plot-45-1.jpg](#), [plot-45-2.jpg](#)
Case 26: [points-50.txt](#), points-50.bin, [cond-50.txt](#), [sol-50.txt](#), [plot-50-1.jpg](#), [plot-50-2.jpg](#)
Case 27: [points-60.txt](#), points-60.bin, [cond-60.txt](#), [sol-60.txt](#), [plot-60-1.jpg](#), [plot-60-2.jpg](#)