

# **Hagen-Poiseuille Flow**

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[http://www.giacomo.lorenzoni.name/PEEI\\_4.0.0.1/Hagen-Poiseuille\\_flow/](http://www.giacomo.lorenzoni.name/PEEI_4.0.0.1/Hagen-Poiseuille_flow/)

[http://www.giacomo.lorenzoni.name/PEEI\\_4.0.0.1/PEEIapplDown.aspx?var=13](http://www.giacomo.lorenzoni.name/PEEI_4.0.0.1/PEEIapplDown.aspx?var=13)

# Hagen-Poiseuille flow

This text is integrating part of the homonymous link in [PEEI: a computer program for the numerical solution of systems of partial differential equations](#).

**System of measurement:** International System of Units

**Coordinate system:** Cylindrical

**Coordinates:**  $\underline{c}$  of which  $\underline{c} \equiv \{c_i; i=1,3\}$   $\mathbb{R}\langle c_1 \rangle \equiv [0, \infty)$   $\mathbb{R}\langle c_2 \rangle \equiv [0, 2\pi)$   $\mathbb{R}\langle c_3 \rangle \equiv (-\infty, \infty)$

**Coordinate versors:**  $\{\mathbf{\kappa}_i; i=1,3\}$

**Unknown functions:**  $\{W_1, W_2, W_3, P\}$  of which  $\mathbf{W} \equiv \sum_{i=1,3} (W_i \cdot \mathbf{\kappa}_i)$ ,  $[W_i] \equiv [\text{speed}]$ ,  $\mathbf{W}$  the velocity vector,  $[P] \equiv [\text{pressure}]$ .

**Deduction of the differential analytical model:**

The  $c_1 = c_1(\underline{x}) \equiv (x_1^2 + x_2^2)^{0.5}$   $c_2 = c_2(\underline{x}) \equiv \arcsin(x_2 / (x_1^2 + x_2^2)^{0.5}) = \arctan\langle x_2 / x_1 \rangle + K$   $c_3 = c_3(\underline{x}) \equiv x_3$  of which  $\underline{x} \equiv \{x_i; i=1,3\}$   $\mathbb{R}\langle x_i \rangle \equiv (-\infty, \infty)$ , imply the  $x_1 = x_1(\underline{c}) \equiv c_1 \cdot \cos\langle c_2 \rangle$   $x_2 = x_2(\underline{c}) \equiv c_1 \cdot \sin\langle c_2 \rangle$   $x_3 = x_3(\underline{c}) \equiv c_3$ , where  $\underline{x}$  are the coordinates of a Cartesian coordinate system that have the versors  $\{\mathbf{v}_i; i=1,3\}$ , and  $K$  is a constant determined by the trigonometric quadrant of  $\{x_1, x_2\}$ . Hence

$$\begin{aligned} \partial c_1 / \partial x_1 &= \cos\langle c_2 \rangle = x_1 / c_1 \quad \partial c_1 / \partial x_2 = \sin\langle c_2 \rangle = x_2 / c_1 \quad \partial c_2 / \partial x_1 = -\sin\langle c_2 \rangle / c_1 = -x_2 / c_1^2 \\ \partial c_2 / \partial x_2 &= \cos\langle c_2 \rangle / c_1 = x_1 / c_1^2 \quad \partial c_1 / \partial x_3 = \partial c_2 / \partial x_3 = \partial c_3 / \partial x_1 = \partial c_3 / \partial x_2 = 0 \quad \partial c_3 / \partial x_3 = 1 \end{aligned}$$

$$\partial_{hi} \equiv \partial c_h / \partial x_i = \hat{o}_{h1i3} \cdot \hat{o}_{h2i3} \cdot \hat{o}_{h3i1} \cdot \hat{o}_{h3i2} \cdot ((\check{o}_{h1i1} \cdot x_1 + \check{o}_{h1i2} \cdot x_2) / c_1 + (\check{o}_{h2i2} \cdot x_1 - \check{o}_{h2i1} \cdot x_2) / c_1^2 + \check{o}_{h3i3})$$

$$\begin{aligned} \partial_{hij}^2 &\equiv \partial^2 c_h / \partial x_i \partial x_j = \hat{o}_{h1i3} \cdot \hat{o}_{h1j3} \cdot \hat{o}_{h2i3} \cdot \hat{o}_{h2j3} \cdot \hat{o}_{h3i1} \cdot \hat{o}_{h3j1} \cdot \hat{o}_{h3i2} \cdot \hat{o}_{h3j2} \cdot \\ &(\check{o}_{j1} \cdot (\check{o}_{h1i1} / c_1 - (\check{o}_{h1i1} \cdot x_1 + \check{o}_{h1i2} \cdot x_2) \cdot x_1 / c_1^3 + \check{o}_{h2i2} / c_1^2 - 2 \cdot (\check{o}_{h2i2} \cdot x_1 - \check{o}_{h2i1} \cdot x_2) \cdot x_1 / c_1^4) + \\ &\check{o}_{j2} \cdot (\check{o}_{h1i2} / c_1 - (\check{o}_{h1i1} \cdot x_1 + \check{o}_{h1i2} \cdot x_2) \cdot x_2 / c_1^3 - \check{o}_{h2i1} / c_1^2 - 2 \cdot (\check{o}_{h2i2} \cdot x_1 - \check{o}_{h2i1} \cdot x_2) \cdot x_2 / c_1^4)) \end{aligned}$$

where  $\check{o}_{ijk} = 1 - \hat{o}_{ijk}$ ,  $\hat{o}_{ijk} = 0$  if  $i=j$   $h=k$  and otherwise  $\hat{o}_{ijk} = 1$ .

The  $\mathbf{\kappa}_i \equiv \sum_{i=1,3} (\mathbf{\kappa}_i \cdot \mathbf{v}_i)$  implies  $\mathbf{\kappa}_i \cdot \mathbf{v}_j = \sum_{i=1,3} (\mathbf{\kappa}_i \cdot \mathbf{v}_i \cdot \mathbf{v}_j)$ . This and  $\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$  (where  $\{\delta_{ij} = 0; \forall i \neq j\}$   $\{\delta_{ij} = 1; \forall i = j\}$ ), imply  $\mathbf{\kappa}_i \cdot \mathbf{v}_j = \kappa_{ij}$  and then  $\kappa_{ij} = \cos\langle \alpha_{ij} \rangle$  where  $\alpha_{ij}$  is the angle between  $\mathbf{\kappa}_i$  and  $\mathbf{v}_j$ . Hence

$$\kappa_{11} = \kappa_{22} = x_1 / c_1 \quad \kappa_{33} = 1 \quad \kappa_{12} = -\kappa_{21} = x_2 / c_1 \quad \kappa_{13} = \kappa_{23} = \kappa_{31} = \kappa_{32} = 0$$

$$\kappa_{ij} = \hat{o}_{i1j3} \cdot \hat{o}_{i2j3} \cdot \hat{o}_{i3j1} \cdot \hat{o}_{i3j2} \cdot ((\check{o}_{i1j1} + \check{o}_{i2j2}) \cdot x_1 / c_1 + (\check{o}_{i1j2} - \check{o}_{i2j1}) \cdot x_2 / c_1 + \check{o}_{i3j3})$$

$$\begin{aligned} \partial_{ijh} &\equiv \partial \kappa_{ij} / \partial x_h = \hat{o}_{i1j3} \cdot \hat{o}_{i2j3} \cdot \hat{o}_{i3j1} \cdot \hat{o}_{i3j2} \cdot (\check{o}_{h1} \cdot ((\check{o}_{i1j1} + \check{o}_{i2j2}) / c_1 - ((\check{o}_{i1j1} + \check{o}_{i2j2}) \cdot x_1 + (\check{o}_{i1j2} - \check{o}_{i2j1}) \cdot x_2) \cdot x_1 / c_1^3) + \\ &\check{o}_{h2} \cdot ((\check{o}_{i1j2} - \check{o}_{i2j1}) / c_1 - ((\check{o}_{i1j1} + \check{o}_{i2j2}) \cdot x_1 + (\check{o}_{i1j2} - \check{o}_{i2j1}) \cdot x_2) \cdot x_2 / c_1^3)) \end{aligned}$$

$$\begin{aligned} \partial^2_{ijk} \equiv \partial^2_{kij} / \partial x_h \partial x_k = & -\hat{o}_{1ij3} \cdot \hat{o}_{2ij3} \cdot \hat{o}_{3ij1} \cdot \hat{o}_{3ij2} \cdot (\bar{\delta}_{k1} \cdot (\bar{\delta}_{h1} \cdot (2 \cdot (\hat{o}_{1ij1} + \hat{o}_{2ij2}) \cdot x_1 / c_1^3 + \\ & ((\hat{o}_{1ij1} + \hat{o}_{2ij2}) \cdot x_1 + (\hat{o}_{1ij2} - \hat{o}_{2ij1}) \cdot x_2) \cdot (c_1^{-3} - 3 \cdot x_1^2 / c_1^5)) + \\ & \bar{\delta}_{h2} \cdot ((\hat{o}_{1ij2} - \hat{o}_{2ij1}) \cdot x_1 / c_1^3 + (\hat{o}_{1ij1} + \hat{o}_{2ij2}) \cdot x_2 / c_1^3 - 3 \cdot ((\hat{o}_{1ij1} + \hat{o}_{2ij2}) \cdot x_1 + (\hat{o}_{1ij2} - \hat{o}_{2ij1}) \cdot x_2) \cdot x_1 \cdot x_2 / c_1^5)) + \\ & \bar{\delta}_{k2} \cdot (\bar{\delta}_{h1} \cdot ((\hat{o}_{1ij1} + \hat{o}_{2ij2}) \cdot x_2 / c_1^3 + (\hat{o}_{1ij2} - \hat{o}_{2ij1}) \cdot x_1 / c_1^3 - 3 \cdot ((\hat{o}_{1ij1} + \hat{o}_{2ij2}) \cdot x_1 + (\hat{o}_{1ij2} - \hat{o}_{2ij1}) \cdot x_2) \cdot x_1 \cdot x_2 / c_1^5)) + \\ & \bar{\delta}_{h2} \cdot (2 \cdot (\hat{o}_{1ij2} - \hat{o}_{2ij1}) \cdot x_2 / c_1^3 + ((\hat{o}_{1ij1} + \hat{o}_{2ij2}) \cdot x_1 + (\hat{o}_{1ij2} - \hat{o}_{2ij1}) \cdot x_2) \cdot (c_1^{-3} - 3 \cdot x_2^2 / c_1^5)))) \end{aligned}$$

The  $\mathbf{v}_i \cdot \mathbf{v}_j = \bar{\delta}_{ij}$   $\mathbf{\kappa}_i \cdot \mathbf{v}_j = \mathbf{\kappa}_{ij}$   $\sum_{j=1,3} (\mathbf{v}_j \cdot \mathbf{\kappa}_j) = \sum_{j=1,3} (\mathbf{v}_j \cdot \mathbf{v}_j)$ , imply

$$\mathbf{v}_i = \sum_{j=1,3} (\mathbf{v}_j \cdot \mathbf{\kappa}_{ji}) \quad (1)$$

From  $F(\underline{x}) \equiv F(\underline{c}(\underline{x}))$  where  $\underline{c}(\underline{x}) \equiv \{c_i(\underline{x}); i=1,3\}$ , follows

$$\partial F(\underline{x}) / \partial x_i = \partial F(\underline{c}(\underline{x})) / \partial x_i = \sum_{i=1,3} ((\partial F(\underline{c}(\underline{x})) / \partial c_i) \cdot (\partial c_i(\underline{x}) / \partial x_i)) \equiv \sum_{i=1,3} ((\partial F / \partial c_i) \cdot \partial_{ii}) \quad (2)$$

$$\begin{aligned} \partial^2 F(\underline{x}) / \partial x_i \partial x_j = & \partial^2 F(\underline{c}(\underline{x})) / \partial x_i \partial x_j = \partial (\partial F(\underline{c}(\underline{x})) / \partial x_i) / \partial x_j = \sum_{i=1,3} (\partial ((\partial F(\underline{c}(\underline{x})) / \partial c_i) \cdot (\partial c_i(\underline{x}) / \partial x_i)) / \partial x_j) = \\ & \sum_{i=1,3} ((\partial F(\underline{c}(\underline{x})) / \partial c_i) \cdot (\partial^2 c_i(\underline{x}) / \partial x_i \partial x_j) + (\partial c_i(\underline{x}) / \partial x_i) \cdot \sum_{j=1,3} ((\partial^2 F(\underline{c}(\underline{x})) / \partial c_i \partial c_j) \cdot (\partial c_j(\underline{x}) / \partial x_j))) = \\ & \sum_{i=1,3} ((\partial F / \partial c_i) \cdot \partial^2_{ii} + \sum_{j=1,3} (\partial_{ii} \cdot \partial_{jj} \cdot (\partial^2 F / \partial c_i \partial c_j))) \end{aligned} \quad (3)$$

The continuity equation for incompressible fluids and the stationary incompressible Navier-Stokes equations for constant viscosity, in Cartesian coordinates are respectively

$$\sum_{i=1,3} (\partial \mathbf{w}_i / \partial x_i) = 0 \quad (4)$$

$$\{\rho \cdot (\sum_{j=1,3} (\mathbf{w}_j \cdot (\partial \mathbf{w}_i / \partial x_j)) - F_i) + \partial P / \partial x_i - \mu \cdot \sum_{j=1,3} (\sum_{h=1,3} (\sum_{k=1,3} (\delta_{jikh} \cdot (\partial^2 \mathbf{w}_k / \partial x_h \partial x_j)))) = 0; i=1,3\} \quad (5)$$

of which:  $\mathbf{w} = \sum_{i=1,3} (\mathbf{w}_i \cdot \mathbf{v}_i)$ ,  $[\mathbf{w}_i] = [\text{speed}]$ ,  $[\rho] = [\text{density}]$ ,  $\mathbf{F} = \sum_{i=1,3} (F_i \cdot \mathbf{v}_i)$ ,  $[F_i] = [\text{force/mass}]$ ,  $\mathbf{F}$  the body force vector per unit mass,  $[\mu] = [\text{dynamic viscosity}]$ ,  $\delta_{ijkh} \equiv \bar{\delta}_{ik} \cdot \bar{\delta}_{jh} + \bar{\delta}_{jk} \cdot \bar{\delta}_{ih} - (2/3) \cdot \bar{\delta}_{hk} \cdot \bar{\delta}_{ij}$ .

From  $\sum_{i=1,3} (\mathbf{w}_i \cdot \mathbf{v}_i) = \sum_{i=1,3} (\mathbf{w}_i \cdot \mathbf{\kappa}_i)$  and (1) follows  $\mathbf{w}_i = \sum_{j=1,3} (\mathbf{w}_j \cdot \mathbf{\kappa}_{ji})$ . This, (4) (5) (2) and (3), imply (6) and (7).

### Differential analytical model:

$$\sum_{i=1,3} (\sum_{j=1,3} (\partial_{jii} \cdot \mathbf{w}_j) + \sum_{h=1,3} (\mathbf{\kappa}_{ji} \cdot \partial_{hi} \cdot (\partial \mathbf{w}_j / \partial c_h))) = 0 \quad (6)$$

$$\begin{aligned} & \{\sum_{j=1,3} (\partial_{jii} \cdot (\partial P / \partial c_j)) + \\ & \sum_{h=1,3} (\sum_{k=1,3} (\rho \cdot \mathbf{\kappa}_{hj} \cdot \partial_{kij} \cdot \mathbf{w}_h \cdot \mathbf{w}_k + \\ & \sum_{m=1,3} (\rho \cdot \mathbf{\kappa}_{hj} \cdot \mathbf{\kappa}_{ki} \cdot \partial_{nj} \cdot \mathbf{w}_h \cdot (\partial \mathbf{w}_k / \partial c_m) - \mu \cdot \delta_{jikh} \cdot \partial^2_{mkhj} \cdot \mathbf{w}_m - \\ & \mu \cdot \delta_{jikh} \cdot \sum_{n=1,3} ((\partial_{nj} \cdot \partial_{mkh} + \partial_{nh} \cdot \partial_{mkj} + \mathbf{\kappa}_{mk} \cdot \partial^2_{nhj}) \cdot (\partial \mathbf{w}_m / \partial c_n)) + \\ & \sum_{p=1,3} (\mathbf{\kappa}_{mk} \cdot \partial_{nh} \cdot \partial_{pj} \cdot (\partial^2 \mathbf{w}_m / \partial c_n \partial c_p)))))) - \rho \cdot F_i = 0; i=1,3\} \end{aligned} \quad (7)$$

of which  $\rho = 998.2071$   $\mu = 0.001003$ . The (6) and (7) are respectively, in cylindrical coordinates, the continuity equation for incompressible fluids and the stationary incompressible Navier-Stokes equations for constant viscosity.

**Known functions:**  $\{F_i; i=1,156\}$  of which  $\{F_i \equiv F_i; i=1,3\}$   $F_1 = F_3 = 0$   $F_2 = -9.80665$   $F_{A(h,i)} = \partial_{hi}$   $F_{B(h,i,j)} = \partial^2_{hij}$   $F_{C(i,j)} = \mathbf{\kappa}_{ij}$   $F_{D(i,j,h)} = \partial_{ijh}$   $F_{E(i,j,h,k)} = \partial^2_{ijkh}$   $A_{hi} = 3 + h + 3 \cdot (i-1)$   $B_{hij} = A_{33} + h + 3 \cdot (i-1) + 9 \cdot (j-1)$   $C_{ij} = B_{333} + i + 3 \cdot (j-1)$   $D_{ijh} = C_{33} + i + 3 \cdot (j-1) + 9 \cdot (h-1)$   $E_{ijkh} = D_{333} + i + 3 \cdot (j-1) + 9 \cdot (h-1) + 27 \cdot (k-1)$ .

**Definition set:**  $\{\underline{c} / R_1 \leq c_1 \leq R_2; 0 \leq c_2 < 2 \cdot \pi; 0 \leq c_3 \leq L_3\}$   $R_1 = 0.001$   $R_2 = 10$   $L_3 = 1000$ .

$$\begin{aligned} \text{Conditions: } W_1(\underline{c})=W_2(\underline{c})=\partial W_3/\partial c_2=\partial W_3/\partial c_3=0 \quad W_3(R_2, c_2, c_3)=0 \quad \partial P(\underline{c})/\partial c_3=K=-1 \\ P(R_2, 0, 0)=100000 \end{aligned} \quad (8)$$

**Related files:** [mad.txt](#)

**Exact solution:** From [here](#) follows

$$W_3=W_3(c_1) \equiv K \cdot (c_1^2 - R_2^2) / (4 \cdot \mu) \quad (9)$$

**Note:** In the following diagrams, the continuous line and the symbol ● (full circle) are respectively inherent to (9) and solution calculated by PEEI.

**Case 1:** [points-1.txt](#), points-1.bin, [cond-1.txt](#), [sol-1.txt](#), [plot-1.jpg](#)

**Case 2:** [points-2.txt](#), points-2.bin, [cond-2.txt](#), [sol-2.txt](#), [plot-2.jpg](#)

**Case 3:** [points-3.txt](#), points-3.bin, [cond-3.txt](#), [sol-3.txt](#), [plot-3.jpg](#)

**Case 4:** [points-4.txt](#), points-4.bin, [cond-4.txt](#), [sol-4.txt](#), [plot-4.jpg](#)

**Case 5:** [points-5.txt](#), points-5.bin, [cond-5.txt](#), [sol-5.txt](#), [plot-5.jpg](#)

**Case 6:** [points-6.txt](#), points-6.bin, [cond-6.txt](#), [sol-6.txt](#), [plot-6.jpg](#)

**Case 7:** [points-7.txt](#), points-7.bin, [cond-7.txt](#), [sol-7.txt](#), [plot-7.jpg](#)

**Case 8:** [points-8.txt](#), points-8.bin, [cond-8.txt](#), [sol-8.txt](#), [plot-8.jpg](#)

**Case 9:** [points-9.txt](#), points-9.bin, [cond-9.txt](#), [sol-9.txt](#), [plot-9.jpg](#)

**Case 10:** [points-10.txt](#), points-10.bin, [cond-10.txt](#), [sol-10.txt](#), [plot-10.jpg](#)

**Case 11:** [points-11.txt](#), points-11.bin, [cond-11.txt](#), [sol-11.txt](#), [plot-11.jpg](#)

**Case 12:** [points-12.txt](#), points-12.bin, [cond-12.txt](#), [sol-12.txt](#), [plot-12.jpg](#)

**Case 13:** [points-13.txt](#), points-13.bin, [cond-13.txt](#), [sol-13.txt](#), [plot-13.jpg](#)

**Case 14:** [points-14.txt](#), points-14.bin, [cond-14.txt](#), [sol-14.txt](#), [plot-14.jpg](#)

**Case 15:** [points-15.txt](#), points-15.bin, [cond-15.txt](#), [sol-15.txt](#), [plot-15.jpg](#)

**Case 16:** [points-16.txt](#), points-16.bin, [cond-16.txt](#), [sol-16.txt](#), [plot-16.jpg](#)

**Case 17:** [points-17.txt](#), points-17.bin, [cond-17.txt](#), [sol-17.txt](#), [plot-17.jpg](#)

**Case 18:** [points-18.txt](#), points-18.bin, [cond-18.txt](#), [sol-18.txt](#), [plot-18.jpg](#)

**Case 19:** [points-19.txt](#), points-19.bin, [cond-19.txt](#), [sol-19.txt](#), [plot-19.jpg](#)

**Case 20:** [points-20.txt](#), points-20.bin, [cond-20.txt](#), [sol-20.txt](#), [plot-20.jpg](#)

**Case 21:** [points-25.txt](#), points-25.bin, [cond-25.txt](#), [sol-25.txt](#), [plot-25.jpg](#)

**Case 22:** [points-30.txt](#), points-30.bin, [cond-30.txt](#), [sol-30.txt](#), [plot-30.jpg](#)

**Case 23:** [points-35.txt](#), points-35.bin, [cond-35.txt](#), [sol-35.txt](#), [plot-35.jpg](#)

**Case 24:** [points-40.txt](#), points-40.bin, [cond-40.txt](#), [sol-40.txt](#), [plot-40.jpg](#)

**Case 25:** [points-45.txt](#), points-45.bin, [cond-45.txt](#), [sol-45.txt](#), [plot-45.jpg](#)

**Case 26:** [points-50.txt](#), points-50.bin, [cond-50.txt](#), [sol-50.txt](#), [plot-50.jpg](#)

**Case 27:** [points-60.txt](#), points-60.bin, [cond-60.txt](#), [sol-60.txt](#), [plot-60.jpg](#)