

Hydrostatic Compression of a Elastic Sphere

16/09/2008

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Hydrostatic compression of a elastic sphere

This text is integrating part of the homonymous link in [PEEI: a computer program for the numerical solution of systems of partial differential equations](#).

System of measurement: International System of Units, with the exception of the force that is expressed in $N \times 10^{-12}$.

Coordinate system: Cartesian

Coordinates: \underline{x} of which: $\underline{x} \equiv \{x_i; i=1,3\}$ $[x_i] = [\text{length}]$ $\mathbb{R}(\underline{x}_i) \equiv (-\infty, \infty)$, \underline{x} a point of the deformed medium.

Coordinate versors: $\{\mathbf{v}_i; i=1,3\}$

Unknown functions: $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \tau_{11}, \tau_{12}, \tau_{13}, \tau_{22}, \tau_{23}, \tau_{33}\}$ of which: $\mathbf{s}_i = \mathbf{x}_i - \mathbf{X}_i$, $[\mathbf{s}_i] = [\text{length}]$, $\underline{X} \equiv \{X_i; i=1,3\}$, \underline{X} the position of the point \underline{x} in the undeformed medium, $\mathbf{s} \equiv \sum_{i=1,3} (\mathbf{s}_i \cdot \mathbf{v}_i)$, \mathbf{s} the displacement of the point \underline{X} , $\{\tau_{11}, \tau_{12}, \tau_{13}, \tau_{22}, \tau_{23}, \tau_{33}\}$ the six independent components of the stress tensor, $[\tau_{ij}] = [\text{stress}]$, $\tau_{ij} = \tau_{ji}$.

Differential analytical model:

$$\partial \tau_{11}(\underline{x}) / \partial x_1 + \partial \tau_{12}(\underline{x}) / \partial x_2 + \partial \tau_{13}(\underline{x}) / \partial x_3 + F_1(\underline{x}) = 0$$

$$\partial \tau_{12}(\underline{x}) / \partial x_1 + \partial \tau_{22}(\underline{x}) / \partial x_2 + \partial \tau_{23}(\underline{x}) / \partial x_3 + F_2(\underline{x}) = 0$$

$$\partial \tau_{13}(\underline{x}) / \partial x_1 + \partial \tau_{23}(\underline{x}) / \partial x_2 + \partial \tau_{33}(\underline{x}) / \partial x_3 + F_3(\underline{x}) = 0$$

$$\{(1+\nu) \cdot \tau_{ij}(\underline{x}) - \delta_{ij} \cdot \nu \cdot (\tau_{11}(\underline{x}) + \tau_{22}(\underline{x}) + \tau_{33}(\underline{x})) - E \cdot (\partial \mathbf{s}_i(\underline{x}) / \partial x_j + \partial \mathbf{s}_j(\underline{x}) / \partial x_i) / 2 = 0; j=i,3; i=1,3\}$$

of which: $\mathbf{F} \equiv \sum_{i=1,3} (F_i \cdot \mathbf{v}_i)$, \mathbf{F} the body force per unit volume, $\{\delta_{ij}=0; \forall i \neq j\}$ $\{\delta_{ij}=1; \forall i=j\}$, E Young's modulus, ν Poisson's ratio, $E=0.21$ $\nu=0.3$.

Related relations:

$$\varepsilon_{ij}(\underline{x}) = (\partial \mathbf{s}_i(\underline{x}) / \partial x_j + \partial \mathbf{s}_j(\underline{x}) / \partial x_i) / 2 = (1+\nu) \cdot \tau_{ij}(\underline{x}) / E - \delta_{ij} \cdot \nu \cdot (\tau_{11}(\underline{x}) + \tau_{22}(\underline{x}) + \tau_{33}(\underline{x})) / E \quad (1)$$

$$\omega_{ij}(\underline{x}) = (\partial \mathbf{s}_i(\underline{x}) / \partial x_j - \partial \mathbf{s}_j(\underline{x}) / \partial x_i) / 2$$

$$\mathbf{s}_i(\underline{x}_B) = \mathbf{s}_i(\underline{x}_A) + \sum_{j=1,3} (\omega_{ij}(\underline{x}_A) \cdot (\mathbf{x}_{Bj} - \mathbf{x}_{Aj})) + \int_{A,B} (\Theta_i(\mathbf{c}) \cdot d\mathbf{c}) \quad (2)$$

$$\Theta_i(\mathbf{c}) \equiv \sum_{j=1,3} (\varepsilon_{ij}(\underline{x}(\mathbf{c})) \cdot \mathbf{x}_j'(\mathbf{c}) + (\mathbf{x}_{Bj} - \mathbf{x}_j(\mathbf{c})) \cdot \sum_{k=1,3} ((\partial \varepsilon_{ik}(\underline{x}(\mathbf{c})) / \partial x_j - \partial \varepsilon_{jk}(\underline{x}(\mathbf{c})) / \partial x_i) \cdot \mathbf{x}_k'(\mathbf{c}))) \quad (3)$$

of which [here](#), $\varepsilon_{ij} = \varepsilon_{ji}$, $\underline{x}(\mathbf{c}) \equiv \{x_i(\mathbf{c}); i=1,3\}$ $\underline{x}_A \equiv \{x_{Ai}; i=1,3\} = \underline{x}(A)$ $\underline{x}_B \equiv \{x_{Bi}; i=1,3\} = \underline{x}(B)$.

Definition set: $\{\underline{x} / x_1^2 + x_2^2 + x_3^2 \leq R^2\}$ $R=1/2$.

Conditions:

$$F_1(\underline{x})=F_2(\underline{x})=F_3(\underline{x})=0 \quad \{\varsigma_i(\underline{x}_A)=0; i=1,3\} \quad \partial\varsigma_1(\underline{x}_A)/\partial x_2=\partial\varsigma_1(\underline{x}_A)/\partial x_3=\partial\varsigma_2(\underline{x}_A)/\partial x_3=0 \quad \underline{x}_A=\{0,0,0\} \quad (4)$$

$$\{\tau_{11}=\tau_{22}=\tau_{33}=-P, \tau_{12}=\tau_{13}=\tau_{23}=0; \forall x_1^2+x_2^2+x_3^2=R^2\} \quad P=0.1 \quad (5)$$

In PEEI executions, the (5) is applied on the surface of the body that approximates the sphere. This is coherent with the validity of this script for a body of every shape.

Related files: [mad.txt](#)

Exact solution:

The (5) and (1) imply

$$\varepsilon_{ij}(\underline{x})=\delta_{ij}\cdot(2\cdot\nu-1)\cdot P/E \quad (6)$$

From these, (1), $\partial\varsigma_i(\underline{x})/\partial x_j=\varepsilon_{ij}(\underline{x})+\omega_{ij}(\underline{x})$ and (4) follows $\omega_{ij}(\underline{x}_A)=0$. This, $\{\varsigma_i(\underline{x}_A)=0; i=1,3\}$ and (2) imply

$$\varsigma_i(\underline{x}_B)=\int_{A,B}(\Theta_i(c)\cdot dc) \quad (7)$$

Are placed

$$\int_{A,B}(\Theta_i(c)\cdot dc)=\int_{A,P}(\Theta_i(c)\cdot dc)+\int_{P,Q}(\Theta_i(c)\cdot dc)+\int_{Q,B}(\Theta_i(c)\cdot dc) \quad \underline{x}(P)=\{x_{B1},0,0\} \quad \underline{x}(Q)=\{x_{B1},x_{B2},0\} \quad \{x_2'(c)=x_3'(c)=0, x_1'(c)=1; \forall c\in[A,P]\} \quad \{x_1'(c)=x_3'(c)=0, x_2'(c)=1; \forall c\in[P,Q]\} \quad \{x_1'(c)=x_2'(c)=0, x_3'(c)=1; \forall c\in[Q,B]\} \quad (8)$$

These, (3) and (6) imply

$$\{\Theta_1(c)=(2\cdot\nu-1)\cdot P/E, \Theta_2(c)=\Theta_3(c)=0; \forall c\in[A,P]\} \quad \{\Theta_2(c)=(2\cdot\nu-1)\cdot P/E, \Theta_1(c)=\Theta_3(c)=0; \forall c\in[P,Q]\} \\ \{\Theta_3(c)=(2\cdot\nu-1)\cdot P/E, \Theta_1(c)=\Theta_2(c)=0; \forall c\in[Q,B]\}$$

From these, (7) and (8) follows

$$\varsigma_i(\underline{x})=(2\cdot\nu-1)\cdot P\cdot x_i/E \quad (9)$$

and hence $\mathbf{s}=\mathbf{s}(\underline{x})=\sum_{i=1,3}(\varsigma_i(\underline{x})\cdot\mathbf{v}_i)=(2\cdot\nu-1)\cdot P\cdot\sum_{i=1,3}(x_i\cdot\mathbf{v}_i)/E$. This imply $\mathbf{s}(\underline{x})=(2\cdot\nu-1)\cdot P\cdot\mathbf{r}(\underline{x})/E$ where $\mathbf{r}(\underline{x})=\sum_{i=1,3}(x_i\cdot\mathbf{v}_i)$; i.e. \mathbf{s} is radial, is directed towards \underline{x}_A , and have magnitude $r(\underline{x})\cdot(1-2\cdot\nu)\cdot P/E$ of which $r(\underline{x})=(\sum_{i=1,3}(x_i^2))^{0.5}$. The (9) is valid for the hydrostatic compression of an elastic body of every shape.

Note: In the following diagrams, the symbols + (plus), □ (empty square) and ■ (full square) are respectively inherent to \underline{x} , \underline{X} determined by means of $X_i=x_i-\varsigma_i$ and (9), and \underline{x} determined by means of $X_i=x_i-\varsigma_i$ where ς_i is calculated by PEEI.

Case 5: [points-5.txt](#), mem-5.bin, [cond-5.txt](#), [sol-5.txt](#), [plot-5-1.jpg](#), [plot-5-2.jpg](#), [plot-5-3.jpg](#), [plot-5-4.jpg](#)

Case 7: [points-7.txt](#), mem-7.bin, [cond-7.txt](#), [sol-7.txt](#), [plot-7-1.jpg](#), [plot-7-2.jpg](#), [plot-7-3.jpg](#), [plot-7-4.jpg](#)

Case 9: [points-9.txt](#), mem-9.bin, [cond-9.txt](#), [sol-9.txt](#), [plot-9-1.jpg](#), [plot-9-2.jpg](#), [plot-9-3.jpg](#), [plot-9-4.jpg](#)

Case 11: [points-11.txt](#), mem-11.bin, [cond-11.txt](#), [sol-11.txt](#), [plot-11-1.jpg](#), [plot-11-2.jpg](#), [plot-11-3.jpg](#), [plot-11-4.jpg](#)

Bibliography:

[1] YU. A. AMENZADE, *Theory of Elasticity*, Mir Publishers, 1979, Moscow