

Elastic torsion of a circular bar

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Giacomo Lorenzoni

http://www.giacomo.lorenzoni.name/PEEI_4.0.0.1/Elastic_torsion_of_a_circular_bar/

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Elastic torsion of a circular bar

This text is integrating part of the homonymous link in [PEEI: a computer program for the numerical solution of systems of partial differential equations](#).

System of measurement: International System of Units, with the exception of the force that is expressed in $N \times 10^{-12}$.

Coordinate system: Cartesian

Coordinates: \underline{x} of which: $\underline{x} \equiv \{x_i; i=1,3\}$ $[x_i] = [\text{length}]$ $\mathcal{R}(\underline{x}_i) \equiv (-\infty, \infty)$, \underline{x} a point of the deformed medium.

Coordinate versors: $\{\mathbf{v}_i; i=1,3\}$

Unknown functions: $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \tau_{11}, \tau_{12}, \tau_{13}, \tau_{22}, \tau_{23}, \tau_{33}\}$ of which: $\mathbf{s}_i = \mathbf{x}_i - \mathbf{X}_i$, $[\mathbf{s}_i] = [\text{length}]$, $\underline{X} \equiv \{X_i; i=1,3\}$, \underline{X} the position of the point \underline{x} in the undeformed medium, $\mathbf{s} \equiv \sum_{i=1,3} (\mathbf{s}_i \cdot \mathbf{v}_i)$, \mathbf{s} the displacement of the point \underline{X} , $\{\tau_{11}, \tau_{12}, \tau_{13}, \tau_{22}, \tau_{23}, \tau_{33}\}$ the six independent components of the stress tensor, $[\tau_{ij}] = [\text{stress}]$, $\tau_{ij} = \tau_{ji}$.

Differential analytical model:

$$\partial \tau_{11}(\underline{x}) / \partial x_1 + \partial \tau_{12}(\underline{x}) / \partial x_2 + \partial \tau_{13}(\underline{x}) / \partial x_3 + F_1(\underline{x}) = 0$$

$$\partial \tau_{12}(\underline{x}) / \partial x_1 + \partial \tau_{22}(\underline{x}) / \partial x_2 + \partial \tau_{23}(\underline{x}) / \partial x_3 + F_2(\underline{x}) = 0$$

$$\partial \tau_{13}(\underline{x}) / \partial x_1 + \partial \tau_{23}(\underline{x}) / \partial x_2 + \partial \tau_{33}(\underline{x}) / \partial x_3 + F_3(\underline{x}) = 0$$

$$\{(1+\nu) \cdot \tau_{ij}(\underline{x}) - \delta_{ij} \cdot \nu \cdot (\tau_{11}(\underline{x}) + \tau_{22}(\underline{x}) + \tau_{33}(\underline{x})) - E \cdot (\partial \mathbf{s}_i(\underline{x}) / \partial x_j + \partial \mathbf{s}_j(\underline{x}) / \partial x_i) / 2 = 0; j=i,3; i=1,3\}$$

of which: $\mathbf{F} \equiv \sum_{i=1,3} (F_i \cdot \mathbf{v}_i)$, \mathbf{F} the body force per unit volume, $\{\delta_{ij}=0; \forall i \neq j\}$ $\{\delta_{ij}=1; \forall i=j\}$, E Young's modulus, ν Poisson's ratio, $E=0.21$ $\nu=0.3$.

Related relations:

$$\varepsilon_{ij}(\underline{x}) = (\partial \mathbf{s}_i(\underline{x}) / \partial x_j + \partial \mathbf{s}_j(\underline{x}) / \partial x_i) / 2 = (1+\nu) \cdot \tau_{ij}(\underline{x}) / E - \delta_{ij} \cdot \nu \cdot (\tau_{11}(\underline{x}) + \tau_{22}(\underline{x}) + \tau_{33}(\underline{x})) / E \quad (1)$$

$$\omega_{ij}(\underline{x}) = (\partial \mathbf{s}_i(\underline{x}) / \partial x_j - \partial \mathbf{s}_j(\underline{x}) / \partial x_i) / 2 \quad (2)$$

$$\mathbf{T}_i(\underline{x}) = \sum_{j=1,3} (\tau_{ji}(\underline{x}) \cdot \mathbf{n}_j(\underline{x})) \quad (3)$$

$$\mathbf{s}_i(\underline{x}_B) = \mathbf{s}_i(\underline{x}_A) + \sum_{j=1,3} (\omega_{ij}(\underline{x}_A) \cdot (\mathbf{x}_{Bj} - \mathbf{x}_{Aj})) + \int_{A,B} (\Theta_i(\mathbf{c}) \cdot d\mathbf{c}) \quad (4)$$

$$\Theta_i(\mathbf{c}) \equiv \sum_{j=1,3} (\varepsilon_{ij}(\underline{x}(\mathbf{c})) \cdot \mathbf{x}_j'(\mathbf{c}) + (\mathbf{x}_{Bj} - \mathbf{x}_j(\mathbf{c})) \cdot \sum_{k=1,3} ((\partial \varepsilon_{ik}(\underline{x}(\mathbf{c})) / \partial x_j - \partial \varepsilon_{jk}(\underline{x}(\mathbf{c})) / \partial x_i) \cdot \mathbf{x}_k'(\mathbf{c}))) \quad (5)$$

of which [here](#), $\varepsilon_{ij} = \varepsilon_{ji}$, $\mathbf{T}(\underline{x}) \equiv \sum_{i=1,3} (\mathbf{T}_i(\underline{x}) \cdot \mathbf{v}_i)$ $\mathbf{n}(\underline{x}) \equiv \sum_{i=1,3} (\mathbf{n}_i(\underline{x}) \cdot \mathbf{v}_i)$, $\mathbf{T}(\underline{x})$ the stress vector in a point of a plane with normal outward versor $\mathbf{n}(\underline{x})$, $\underline{x}(\mathbf{c}) \equiv \{x_i(\mathbf{c}); i=1,3\}$ $\underline{x}_A \equiv \{x_{Ai}; i=1,3\} = \underline{x}(A)$ $\underline{x}_B \equiv \{x_{Bi}; i=1,3\} = \underline{x}(B)$

Definition set: $\{\underline{x} / x_1^2 + x_3^2 \leq R^2; 0 \leq x_2 \leq L_2\}$ $R=1/2$ $L_2=10$.

Conditions:

$$F_1(\underline{x})=F_2(\underline{x})=F_3(\underline{x})=0 \quad \{\varsigma_i(x_1,0,x_3)=0; i=1,3\} \quad \partial \varsigma_1(\underline{x}_A)/\partial x_2=\partial \varsigma_3(\underline{x}_A)/\partial x_2=0 \quad \underline{x}_A \equiv \{0,0,0\} \quad (6)$$

$$\begin{aligned} \tau_{11}(x_1,0,x_3)=\mu \cdot \alpha \cdot x_3 \quad \tau_{21}(x_1,0,x_3)=0 \quad \tau_{31}(x_1,0,x_3)=-\mu \cdot \alpha \cdot x_1 \quad \tau_{11}(x_1,L_2,x_3)=-\mu \cdot \alpha \cdot x_3 \quad \tau_{21}(x_1,L_2,x_3)=0 \\ \tau_{31}(x_1,L_2,x_3)=\mu \cdot \alpha \cdot x_1 \quad \mu=E/(2 \cdot (1+\nu)) \quad \alpha=1 \end{aligned} \quad (7)$$

In PEEI executions, the (7) is applied on the surface of the body that approximates the circular bar. This is coherent with the validity of this script for a bar with the cross section of every shape.

From (7) and (3) follows

$$\tau_{12}(x_1,0,x_3)=\tau_{12}(x_1,L_2,x_3)=-\mu \cdot \alpha \cdot x_3 \quad \tau_{22}(x_1,0,x_3)=\tau_{22}(x_1,L_2,x_3)=0 \quad \tau_{23}(x_1,0,x_3)=\tau_{23}(x_1,L_2,x_3)=\mu \cdot \alpha \cdot x_1$$

Related files: [mad.txt](#)

Exact solution:

From previous conditions follows $\tau_{11}(\underline{x})=\tau_{13}(\underline{x})=\tau_{22}(\underline{x})=\tau_{33}(\underline{x})=0$ $\tau_{12}(\underline{x})=-\mu \cdot \alpha \cdot x_3$ $\tau_{23}(\underline{x})=\mu \cdot \alpha \cdot x_1$. These and (1) imply

$$\varepsilon_{11}(\underline{x})=\varepsilon_{22}(\underline{x})=\varepsilon_{33}(\underline{x})=\varepsilon_{13}(\underline{x})=0 \quad \varepsilon_{12}(\underline{x})=-\alpha \cdot x_3/2 \quad \varepsilon_{23}(\underline{x})=\alpha \cdot x_1/2 \quad (8)$$

From (2) and (6) follows $\omega_{ij}(\underline{x}_A)=0$. This, $\{\varsigma_i(\underline{x}_A)=0; i=1,3\}$ and (4) imply

$$\varsigma_i(\underline{x}_B)=\int_{A,B}(\Theta_i(c) \cdot dc) \quad (9)$$

Are placed

$$\begin{aligned} \int_{A,B}(\Theta_i(c) \cdot dc)=\int_{A,P}(\Theta_i(c) \cdot dc)+\int_{P,Q}(\Theta_i(c) \cdot dc)+\int_{Q,B}(\Theta_i(c) \cdot dc) \quad \underline{x}(P) \equiv \{0, x_{B2}, 0\} \quad \underline{x}(Q) \equiv \{x_{B1}, x_{B2}, 0\} \\ \{x_1'(c)=x_3'(c)=0, x_2'(c)=1; \forall c \in [A,P]\} \quad \{x_2'(c)=x_3'(c)=0, x_1'(c)=1; \forall c \in [P,Q]\} \\ \{x_1'(c)=x_2'(c)=0, x_3'(c)=1; \forall c \in [Q,B]\} \end{aligned} \quad (10)$$

These, (5) and (8) imply

$$\begin{aligned} \{\Theta_1(c)=\alpha \cdot (x_3(c)/2 - x_{B3}), \Theta_2(c)=0, \Theta_3(c)=\alpha \cdot (x_{B1} - x_1(c)/2); \forall c \in [A,P]\} \\ \{\Theta_1(c)=0, \Theta_2(c)=-\alpha \cdot x_{B3}/2, \Theta_3(c)=\alpha \cdot (x_{B2} - x_2(c))/2; \forall c \in [P,Q]\} \\ \{\Theta_1(c)=-\alpha \cdot (x_{B2} - x_2(c))/2, \Theta_2(c)=\alpha \cdot x_{B1}/2, \Theta_3(c)=0; \forall c \in [Q,B]\} \end{aligned}$$

From these, (9) and (10) follows

$$\varsigma_1(\underline{x})=-\alpha \cdot x_2 \cdot x_3 \quad \varsigma_2(\underline{x})=0 \quad \varsigma_3(\underline{x})=\alpha \cdot x_1 \cdot x_2 \quad (11)$$

The (11) is valid for the elastic torsion of a bar with the cross section of every shape.

Note: In the following diagrams, the symbols + (plus), □ (empty square) and ■ (full square) are respectively inherent to \underline{x} , \underline{X} determined by means of $X_i=x_i-\varsigma_i$ and (11), and \underline{x} determined by means of $X_i=x_i-\varsigma_i$ where ς_i is calculated by PEEI.

Case 5-3-5: [points-5-3-5.txt](#), [mem-5-3-5.bin](#), [cond-5-3-5.txt](#), [sol-5-3-5.txt](#),
[plot-5-3-5-1.jpg](#), [plot-5-3-5-2.jpg](#), [plot-5-3-5-3.jpg](#)

Case 5-6-5: [points-5-6-5.txt](#), [mem-5-6-5.bin](#), [cond-5-6-5.txt](#), [sol-5-6-5.txt](#),
[plot-5-6-5-1.jpg](#), [plot-5-6-5-2.jpg](#), [plot-5-6-5-3.jpg](#)

Case 5-9-5: [points-5-9-5.txt](#), [mem-5-9-5.bin](#), [cond-5-9-5.txt](#), [sol-5-9-5.txt](#),
[plot-5-9-5-1.jpg](#), [plot-5-9-5-2.jpg](#), [plot-5-9-5-3.jpg](#)

Case 5-11-5: [points-5-11-5.txt](#), [mem-5-11-5.bin](#), [cond-5-11-5.txt](#), [sol-5-11-5.txt](#),
[plot-5-11-5-1.jpg](#), [plot-5-11-5-2.jpg](#), [plot-5-11-5-3.jpg](#)

Case 7-3-7: [points-7-3-7.txt](#), [mem-7-3-7.bin](#), [cond-7-3-7.txt](#), [sol-7-3-7.txt](#),
[plot-7-3-7-1.jpg](#), [plot-7-3-7-2.jpg](#), [plot-7-3-7-3.jpg](#)

Case 7-6-7: [points-7-6-7.txt](#), [mem-7-6-7.bin](#), [cond-7-6-7.txt](#), [sol-7-6-7.txt](#),
[plot-7-6-7-1.jpg](#), [plot-7-6-7-2.jpg](#), [plot-7-6-7-3.jpg](#)

Case 7-9-7: [points-7-9-7.txt](#), [mem-7-9-7.bin](#), [cond-7-9-7.txt](#), [sol-7-9-7.txt](#),
[plot-7-9-7-1.jpg](#), [plot-7-9-7-2.jpg](#), [plot-7-9-7-3.jpg](#)

Case 7-11-7: [points-7-11-7.txt](#), [mem-7-11-7.bin](#), [cond-7-11-7.txt](#), [sol-7-11-7.txt](#),
[plot-7-11-7-1.jpg](#), [plot-7-11-7-2.jpg](#), [plot-7-11-7-3.jpg](#)

Case 9-3-9: [points-9-3-9.txt](#), [mem-9-3-9.bin](#), [cond-9-3-9.txt](#), [sol-9-3-9.txt](#),
[plot-9-3-9-1.jpg](#), [plot-9-3-9-2.jpg](#), [plot-9-3-9-3.jpg](#)

Case 9-6-9: [points-9-6-9.txt](#), [mem-9-6-9.bin](#), [cond-9-6-9.txt](#), [sol-9-6-9.txt](#),
[plot-9-6-9-1.jpg](#), [plot-9-6-9-2.jpg](#), [plot-9-6-9-3.jpg](#)

Case 9-9-9: [points-9-9-9.txt](#), [mem-9-9-9.bin](#), [cond-9-9-9.txt](#), [sol-9-9-9.txt](#),
[plot-9-9-9-1.jpg](#), [plot-9-9-9-2.jpg](#), [plot-9-9-9-3.jpg](#)

Case 9-11-9: [points-9-11-9.txt](#), [mem-9-11-9.bin](#), [cond-9-11-9.txt](#), [sol-9-11-9.txt](#),
[plot-9-11-9-1.jpg](#), [plot-9-11-9-2.jpg](#), [plot-9-11-9-3.jpg](#)

Bibliography:

[1] YU. A. AMENZADE, *Theory of Elasticity*, Mir Publishers, 1979, Moscow